NTNU

Department of Mathematical Sciences

Examination paper for **TMA4205 Numerical Linear Algebra**

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Permitted examination support material: C: Specified, written and handwritten examination support materials are permitted. A specified, simple calculator is permitted. The permitted examination support materials are:

- Y. Saad: Iterative Methods for Sparse Linear Systems. 2nd ed. SIAM, 2003 (book or printout)
- L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997 (book or photocopy)
- G. Golub and C. Van Loan: Matrix Computations. 3rd ed. The Johns Hopkins University Press, 1996 (book or photocopy)
- J. W. Demmel: Applied Numerical Linear Algebra, SIAM, 1997 (book or printout)
- E. Rønquist: Note on The Poisson problem in $\mathbb{R}^2$: diagonalization methods (printout)
- K. Rottmann: Matematisk formelsamling
- Your own lecture notes from the course (handwritten)

Language: English

Number of pages: 3

Number pages enclosed: 0

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Date Signature

Problem 1

a) Let

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}.$$

Perform one iteration of QR-algorithm (for computing eigenvalues) with the shift parameter $\mu = 1$.

b) Let now $A \in \mathbb{R}^{n\times n}$ be an arbitrary square matrix. Assume that the shift μ in the QR-algorithm with shifts is equal to one of the eigenvalues of A. How can we easily detect this situation based on the QR factorization of the shifted matrix?

Problem 2

Let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

a) Find a singular value decomposition of A.

b) Find the best, with respect to the $\|\cdot\|_2$-norm, rank 1 approximation of A^{-1}. That is, find some vectors $p, q \in \mathbb{R}^2$ minimizing the norm $\|pq^{\mathrm{T}} - A^{-1}\|_2$.

c) Let $B \in \mathbb{R}^{n\times n}$ be an arbitrary non-singular matrix, and let $x_0, b \in \mathbb{R}^n$ be given vectors. Further, let $B_1 = pq^{\mathrm{T}}$ be a rank 1 approximation of B^{-1} for some $p, q \in \mathbb{R}^n$. Consider a general projection method with a search space $\mathcal{K}$ and a constraint space $\mathcal{L}$ for solving a left-preconditioned linear algebraic system $B_1 B x = B_1 b$. Show that $\tilde{x} \in x_0 + \mathcal{K}$ satisfies Petrov–Galerkin conditions if and only if at least one of the following conditions hold:

$$(i) \quad b - B\tilde{x} \perp q \qquad \text{or} \qquad (ii) \quad p \perp \mathcal{L}.$$

Problem 3 Let $A \in \mathbb{C}^{n\times n}$ be an arbitrary square matrix. We put $B = (A + A^{\mathrm{H}})/2$, $C = (A - A^{\mathrm{H}})/2$; in particular $A = B + C$.

a) Show that $\sigma(B) \subset \mathbb{R}$ and $\sigma(C) \subset i\mathbb{R}$, where $\sigma(\cdot)$ is the spectrum of a matrix and $i^2 = -1$.

b) Let $\alpha \in \mathbb{C}$ be an arbitrary scalar, and $I \in \mathbb{R}^{n\times n}$ be the identity matrix. Show that $B \pm \alpha I$ and $C \pm \alpha I$ are unitarily diagonalizable.

Assume now that that both $B + \alpha I$ and $C + \alpha I$ are non-singular. Consider the following iterative algorithm for solving the system $Ax = b$ starting with some initial approximaiton $x_0 \in \mathbb{C}^n$:

for $k = 0, 1, \dots,$ **do**
  $x_{k+1/2} = (B + \alpha I)^{-1}[b - (C - \alpha I)x_k]$
  $x_{k+1} = (C + \alpha I)^{-1}[b - (B - \alpha I)x_{k+1/2}]$
end for

c) Show that if $\alpha = 0$ the algorithm converges, but the limits $\bar{x} = \lim_{k\to\infty} x_k$ and $\hat{x} = \lim_{k\to\infty} x_{k+1/2}$ do not necessarily coincide or solve the system $A\bar{x} = b$ (or $A\hat{x} = b$).

d) Show that the sequence of points x_k generated by the algorithm satisfies $\lim_{k\to\infty} \|x_k - A^{-1}b\|_2 = 0$ *for an arbitrary* $x_0 \in \mathbb{C}^n$ if and only if $\rho((C + \alpha I)^{-1}(B - \alpha I)(B + \alpha I)^{-1}(C - \alpha I)) < 1$, where $\rho(\cdot)$ is a spectral radius of a matrix.

e) Let A be Hermitian and positive definite. Show that the algorithm above converges for an arbitrary $\alpha > 0$ and $x_0 \in \mathbb{C}$.

Problem 4 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix and $b \in \mathbb{R}^n$ be an arbitrary vector. Let $x_* = A^{-1}b$.

a) Show that the standard A-norm error estimate for the conjugate gradient algorithm:

$$\|x_m - x_*\|_A \le 2\left[\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right]^m \|x_0 - x_*\|_A,$$

implies the 2-norm estimate

$$\|x_m - x_*\|_2 \le 2\sqrt{\kappa}\left[\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right]^m \|x_0 - x_*\|_2,$$

where $\kappa = \lambda_{\max}(A)/\lambda_{\min}(A)$ is the spectral condition number of A.

Let V_{2m}, H_{2m} be the matrices produced after $2m$ iterations of Arnoldi process (without breakdown) applied to A, starting from some vector $r_0 = b - Ax_0$.

Let $\hat{V}_m := [V_{*,1}, V_{*,3}, \dots, V_{*,2m-1}]$ (i.e., the submatrix of V_{2m} corresponding to *odd* columns), and $\hat{H}_m = \hat{V}_m^{\mathrm{T}} A \hat{V}_m$.

b) Show that $\hat{H}_m$ is a non-singular diagonal matrix.

c) Consider now an orthogonal projection method for the system $Ax = b$ with $\mathcal{L} = \mathcal{K} = \mathrm{Ran}(\hat{V}_m)$, where we seek $\hat{x}_m \in x_0 + \mathcal{K}$ satisfying the Galerkin orthogonality condition. Show that $\hat{x}_m = x_0 + (r_0^{\mathrm{T}} r_0)(r_0^{\mathrm{T}} A r_0)^{-1} r_0$. (That is, the method takes one steepest descent step and then stops improving the solution.)