Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2015

Exercise set 1

1 We say that two norms $\|\cdot\|_\alpha$ og $\|\cdot\|_\beta$ are equivalent in $\mathbb{C}^n$ if there exist positive constants c_1 and c_2 independent of x, such that

$$c_1\|x\|_\alpha \le \|x\|_\beta \le c_2\|x\|_\alpha. \tag{1}$$

Show that $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent in $\mathbb{C}^n$, by showing that for any vector $x \in \mathbb{C}^n$,

a) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$

b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$

c) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

d) Use the inequalities **a)–c)** to establish equivalence of *matrix* norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ on $\mathbb{C}^{n\times n}$ and estimate the equivalence constants c_1, c_2 (see (1)).

2 Let $\|\cdot\|$ be a vector norm on $\mathbb{C}^n$ and consider the corresponding induced (or natural) matrix norm $\|\cdot\|$. Show that $\rho(A) \le \|A\|$ where $\rho(A)$ is the spectral radius of A.

3 Suppose that $E = uv^{\mathrm{H}}$ is the outer product (or tensor product) of two vectors $u, v \in \mathbb{C}^n$.

a) Show that $\|E\|_2 = \|u\|_2\|v\|_2$.

b) Decide if this also holds for the Frobenius norm, i.e. if $\|E\|_{\mathrm{F}} = \|u\|_2\|v\|_2$.

4 What can be said about the eigenvalues of a unitary matrix?

5 A matrix $A \in \mathbb{C}^{m\times n}$, $m \ge n$, is said to have full rank if its columns are linearly independent, i.e. $c_1 a_1 + \cdots + c_n a_n = 0 \implies c_1 = \cdots = c_n = 0$. Show that A has full rank if, and only if, no two distinct vectors are mapped to the same vector.

6 Consider the matrix-matrix product $B = AR$ where A is an $m \times n$ matrix, and R is an $n \times n$ matrix with elements $r_{i,j} = 1$ for $i \le j$ and $r_{i,j} = 0$ for $i > j$. Show that column j in the matrix B can be written as a sum of the first j columns of A.

7 We will here investigate how the storage format and structure of a matrix influence the performance of the LU factorization of the matrix. MATLAB has two storage formats for matrices. We can either store them as full matrices, that is, all elements of the matrix are stored, or we can store them as sparse matrices where only non-zero elements are stored.

The commands

```
F = full(S)
S = sparse(F)
```

convert a sparse matrix S into a full matrix F and a full matrix F into a sparse matrix S respectively. To depict the non-zero elements of A one may use the following command

```
spy(A)
```

a) We here consider the one-dimensional Poisson problem

$$\begin{aligned} -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= 4\pi^2 \sin(2\pi x), \quad && x \in [0,1], \\ u &= 0, && x \in \{0,1\}. \end{aligned}$$

The MATLAB file `poisson1.m`, which can be fetched from the home page of the course, generates the system of linear equations obtained from discretizing the problem with a finite difference method on a uniform grid. For instance, the command

```
[A, b] = poisson1(n)
```

will return the system of equations having n unknowns. The matrix A will here be stored as a full matrix.

i) For $n = 900, 1600, 2500, 3600$, generate the system of linear equations and measure the time it takes to solve the system with Gaussian elimination (i.e. with LU factorization).

```
[A, b] = poisson1(n)
tic; [L, U] = lu(A); x = U \ (L \ b); toc
```

ii) Repeat the experiment above, but convert A to a sparse matrix before the system is solved. Compare with **i)** and try to explain the difference.

b) We now consider the two-dimensional Poisson problem

$$\begin{aligned} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) &= 5\pi^2 \sin(2\pi x)\sin(\pi y), \qquad && (x,y) \in [0,1]\times[0,1], \\ u &= 0, && x=0, x=1, y=0, y=1. \end{aligned}$$

The MATLAB file `poisson2.m` generates the system of linear equations we obtain when we discretize the above problem with a finite difference method. The command

```
[A, b] = poisson2(n)
```

will generate a system with $N = n^2$ unknowns.

i) For $n = 30, 40, 50, 60$, generate the system of linear equations and measure the time it takes to solve the system with Gaussian elimination. Compare with **a)**.

ii) Repeat step **i)**, but convert A into a sparse matrix before solving the system. Compare with **a)**. Check the structure of the matrices before and after Gaussian elimination in **a)** and **b)**.

```
spy(A)
[L, U] = lu(A)
spy(L); spy(U)
```