Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra
Fall 2015

Exercise set 5

1 Implement a Matlab program for finding an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, v)$ in a "naïve" fashion. That is, given A, v, first generate an $n \times m$ matrix K with $K_{*1} = v/\|v\|_2$ and $K_{*i} = AK_{*i-1}/\|AK_{*i-1}\|_2$ of normalized vectors, which span $\mathcal{K}_m(A, v)$. Then find a "tall" QR-factorization of $K = V_m R_m$; this produces an orthogonal basis $V_m = [v_1, \ldots, v_m]$ for $\mathcal{K}_m(A, v)$.

Use the matrix A generated by `poisson2.m` (use sparse matrices). Take $N = n^2 = 100$, $m = 50$, $v = \mathrm{e}_1$.

Compute the rank of the matrices K and R_m; do they predict the dimension of $\mathcal{K}_m(A, v)$ correctly? Plot the absolute values of the diagonal elements in R on a logarithmic scale.

2 Algorithm 6.1 in Saad is implemented in the MATLAB-function `arnoldi_gs.m`. This algorithm constructs an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, v)$ based on a classical Gram–Schmidt procedure. Test this function on the matrix A generated by `poisson2.m` (use sparse matrices) for different values of m and $N = n^2$. For instance, choose $N = 100$, $v = \mathrm{e}_1$, and $m = 10, 20, 30, 40$.

a) Test to what extent the relation $V_m^{\mathrm{T}} A V_m = H_m$ from Proposition 6.5 in Saad is fulfilled. Also check if the vectors $v_1, \ldots, v_m$ really are orthonormal, i.e. check whether $V_m^{\mathrm{T}} V_m = \mathrm{I}_m$.

b) Modify the function `arnoldi_gs.m` such that it uses modified Gram–Schmidt. Repeat the experiments from the previous question.

Hint: plot the norms of the columns of the matrices $V_m^{\mathrm{T}} V_m - I_m$, $AV_m - V_{m+1}\bar{H}_m$, $V_m^{\mathrm{T}} A V_m - H_m$, vs. $i = 1, \ldots m$

3 Implement Householder reflection-based Arnoldi iteration `arnoldi_h.m` in Matlab with the same interface as `arnoldi_gs.m` (see Algorithm 6.3 in Saad). Repeat the numerical experiment of the exercise 2a) and compare the results.

4 (≈Exercise 6.8 in Saad.) Compute the matrices V_m, H_m, $m = 1, \ldots, 5$ resulting from the application of Arnoldi process to

$$A = \begin{pmatrix} &&&&1\\ 1&&&&\\ &1&&&\\ &&1&&\\ &&&1& \end{pmatrix}, \quad b = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \quad \text{and} \quad x_0 = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}.$$

Compute the FOM iterates y_m, x_m, $m = 1, \ldots, m$ (when possible).