Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2015

Exercise set 6

$\boxed{1}$ (≈Exercise 6.8 in Saad.)

a) Compute the matrices V_m, $\bar{H}_m$, $m = 1, \dots, 5$ resulting from the application of Arnoldi process to

$$A = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

b) Describe in detail the QR-factorization of $\bar{H}_m$, $m = 1, \dots, 5$, using Givens rotations.

c) Compute the GMRES iterates y_m, x_m, $m = 1, \dots, m$ (when possible).

$\boxed{2}$ (Saad, Exercise 6.4) Consider a variant of the GMRES algorithm in which the Arnoldi process starts with $v_1 = Av_0/\|Av_0\|_2$, where $v_0 = r_0$. The Arnoldi process is performed the same way as before to build an orthonormal system $v_1, v_2, \dots, v_{m-1}$. Now the approximate solution is expressed in the basis $\{v_0, v_1, \dots, v_{m-1}\}$.

a) Show that the least-squares problem that must be solved to obtain the approximate solution is now triangular instead of Hessenberg.

b) Show that the residual vector r_m is orthogonal to $v_1, v_2, \dots, v_m$.

c) Find a formula that computes the residual norm (without computing the approximate solution) and write the complete algorithm.

$\boxed{3}$ Assume that a real matrix A is anti-symmetric, that is, $A^{\mathrm{T}} = -A$. Explain the structure of the Hessenberg matrix H_m resulting from Arnoldi process in this case. Explain how this structure can be utilized for performing Arnoldi process efficiently in this case.

$\boxed{4}$ Show that both CG and GMRES are scaling independent. That is, when applied to a system $(\delta A)x = \delta b$ for some $\delta \in \mathbb{R} \setminus \{0\}$ ($\delta > 0$ in the case of CG) they produce exactly the same sequence of iterates in exact arithmetics. (Of course since the residual vectors will be scaled by δ this may affect stopping criteria of the methods.)