Error estimates for MINRES.

MINRES is equivalent to GMRES / ~~DQ~~GMRES applied to a symmetric m-x A

(Recall: in this case Arnoldi process reduces to Lanczos 3 term recursion)

$$\|r_m\|_2 = \min_{x \in x_0 + K_m(A, r_0)} \|b - Ax\|_2$$

$$= \min_{\delta \in K_m(A, r_0)} \|r_0 - A\delta\|_2 = \min_{p_{m-1} \in \mathbb{P}_{m-1}} \|r_0 - A p_{m-1}(A) r_0\|_2$$

$$= \min_{\substack{\tilde{p}_m \in \mathbb{P}_m \\ \tilde{p}_m(0) = 1}} \|\tilde{p}_m(A) r_0\|_2$$

A is normal $\Rightarrow$ $\forall$ eigenvalues are real, eigenvectors can be selected to form an orthonormal basis

$$A = X^T \Lambda X \quad , \quad \Lambda = \operatorname{diag}(\lambda_i)$$

$\Rightarrow$

$$\|r_m\|_2 \le \min_{\substack{\tilde{p}_m \in \mathbb{P}_m \\ \tilde{p}_m(0) = 1}} \max_{1 \le i \le n} |\tilde{p}_m(\lambda_i)| \cdot \|r_0\|_2$$

Proposition: $A = A^T$, has k distinct eigenvalues $\neq 0$

Then MINRES converges in at most k iterations.

Proof: Take $\tilde{p}_k(\lambda) = \prod_{i=1}^{k} (\lambda_i - \lambda) \Big/ \prod_{i=1}^{k} \lambda_i$ ∎

If $\forall$ eigenvalues are positive $\Rightarrow$ same error estimate as for CG [But for residuals, not errors]:

$$\frac{\|r_m\|_2}{\|r_0\|_2} \leq \left(\frac{\sqrt{\lambda_{max}} - \sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}} + \sqrt{\lambda_{min}}} \right)^m \sim \left(1 - \frac{2}{\sqrt{\kappa}} \right)^m$$

for $\lambda_{max} \gg \lambda_{min}$

where $\kappa = \lambda_{max} / \lambda_{min}$ – condition number (spectral)

Same story for only negative eigenvalues!

Again, the estimate is obtained by considering an affine map $t(\lambda) : [\lambda_{min}, \lambda_{max}] \to [-1, 1]$

and then taking $\tilde{p}_m(\lambda) = \dfrac{C_m(t(\lambda))}{C_m(t(0))}$

$$|\tilde{p}_m(\lambda)| \leq \frac{1}{|C_m(t(0))|} \quad \forall\ \lambda_{min} \leq \lambda \leq \lambda_{max}$$

then $|C_m(t(0))| = \left| C_m\left(\dfrac{\lambda_{min} + \lambda_{max}}{\lambda_{max} - \lambda_{min}} \right) \right|$ is estimated from below!

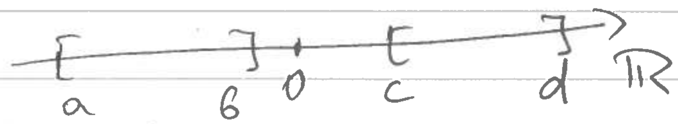
(see pp 204–205 in Saad)

If A is indefinite (has both positive & negative eigenvalues) $\Rightarrow$

$$\min_{\substack{\tilde{p}_m \in \mathbb{P}_m \\ \tilde{p}_m(0)=1}} \max_{\lambda_{min} \le \lambda \le \lambda_{max}} |\tilde{p}_m(\lambda)| = 1,$$ attained by $\tilde{p}_m(\lambda) \equiv 1$

Thus this estimate is useless! Need a diff. idea!

$\lambda_i \in [a, b] \cup [c, d] \not\ni 0$



$a < b < 0 < c < d$, $\underline{\underline{b - a = d - c = L}}$

E.g. can take

$$\tilde{p}_m(\lambda) = \frac{C_k(t(\lambda))}{C_k(t(0))}$$

where $k = \lfloor \frac{m}{2} \rfloor$

C_k – Chebyshev polynomial

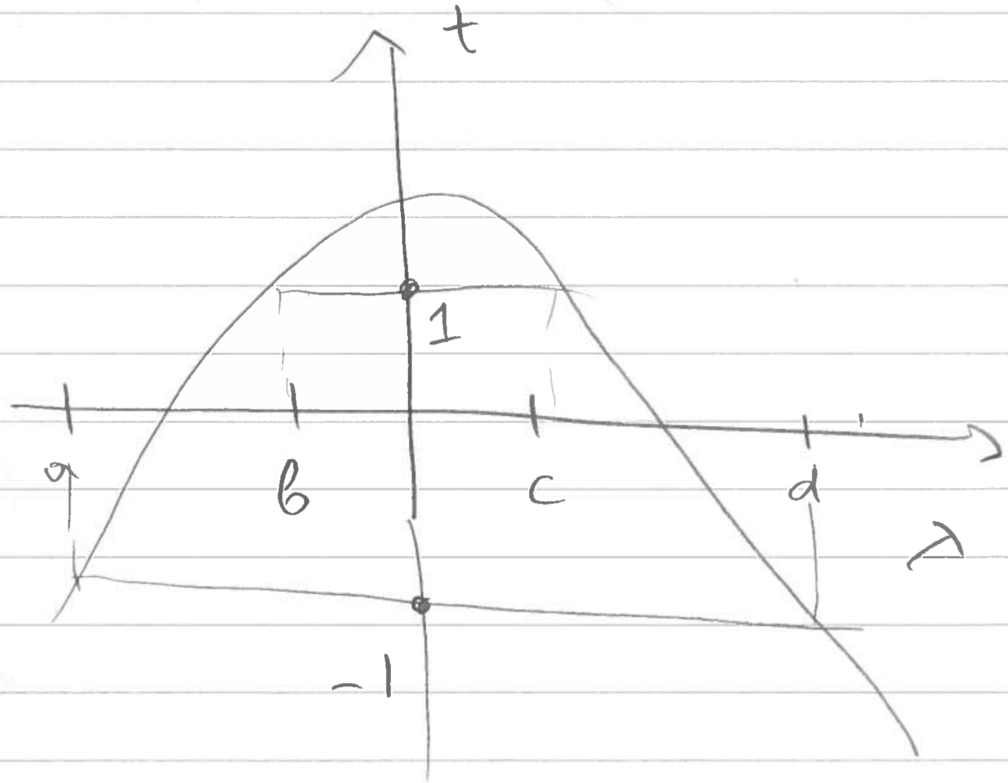
$t : \mathbb{R} \to \mathbb{R}$ is a <u>quadratic</u> mapping

such that $t([a, b] \cup [c, d]) \subset [-1, 1]$!

Require: $t(\lambda)$ - quadratic polynomial

$$t(b) = t(c) = 1$$

$$t(a) = t(b-L) = t(d) = t(c+L) = -1$$



$$t(\lambda) = 1 + \frac{2(\lambda-b)(\lambda-c)}{ad-bc}$$

Clearly $t(b) = t(c) = 1$

Is $t(a) = -1$?

$$d = c+L \Rightarrow ad = ac + aL$$

$$b = a+L \Rightarrow bc = ac + cL$$

$$\Downarrow$$

$$ad - bc = L(a-c)$$

$$t(a) = 1 + \frac{2(a-b)(a-c)}{ad-bc} = 1 + \frac{2(-L)(a-c)}{L(a-c)} = -1$$

Similarly, $t(d) = 1 + \dfrac{2(d-b)(d-c)}{ad - bc}$

$$= 1 + \frac{2(d-b)L}{(d-b)(-L)} = -1$$

because

$$a = b - L \Rightarrow ad = db - dL$$
$$c = d - L \Rightarrow bc = bd - bL$$
$$\Downarrow$$
$$ad - bc = (d-b)(-L)$$

Therefore

$$\max_{\lambda \in [a,b] \cup [c,d]} |\tilde{p}_m(\lambda)| \leq \max_{\lambda \in [a,b] \cup [c,d]} \frac{|C_k(t(\lambda))|}{|C_k(t(0))|}$$

$$\leq \frac{1}{|C_k(t(0))|} = \frac{1}{\left|C_k\left(1 + 2\underbrace{\frac{bc}{ad-bc}}_{\eta > 0}\right)\right|}$$

Following pp 204-205 in Saad we get:

$$C_k(1+2\eta) \geq \frac{1}{2}\left(\sqrt{\eta} + \sqrt{\eta+1}\right)^{2k}$$

$$= \frac{1}{2}\left(\sqrt{\frac{bc}{ad-bc}} + \sqrt{\frac{ad}{ad-bc}}\right)^{2k}$$

$$\frac{1}{2}\left[\frac{\left(\sqrt{|bc|} + \sqrt{|ad|}\right)^2}{|ad| - |bc|}\right]^k = \frac{1}{2}\left[\frac{\sqrt{|bc|} + \sqrt{|ad|}}{\sqrt{|ad|} - \sqrt{|bc|}}\right]^k$$

Therefore, after m iter. of MINRES:

$$\frac{\|r_m\|_2}{\|r_0\|_2} \le 2\left(\frac{\sqrt{|ad|} - \sqrt{|bc|}}{\sqrt{|ad|} + \sqrt{|bc|}}\right)^{\lfloor m/2 \rfloor}$$

Special case: symmetric distribution of eigenvalues around 0

$(a = -d, \quad b = -c)$

$$\Rightarrow \quad \frac{\|r_m\|_2}{\|r_0\|_2} \le 2\left(\frac{d-c}{d+c}\right)^{\lfloor m/2 \rfloor}$$

$$= 2\left(\frac{d/c - 1}{d/c + 1}\right)^{\lfloor m/2 \rfloor} \approx 2\left(1 - \frac{2}{\kappa}\right)^{\lfloor m/2 \rfloor}$$

$d/c = \kappa$ - spectral condition number

Note: in this case need 2x as many iterations as for the positive definite m-x with $d^2/c^2 = \kappa^2$ condition number!