Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam
Name: Brynjulf Owren (93021641)

EXAM IN NUMERICAL LINEAR ALGEBRA (TMA4205)

Friday December 5, 2008
Time: 09:00–13:00

Aids: Category A, All printed and hand written aids allowed. All calculators allowed.

Problem 1 The partial differential equation

$$-u_{xx} + c\,u = f, \quad c \geq 0.$$

with homogeneous Dirichlet boundary conditions, yields after discretizing with centered differences, a linear system of the form $Au = b$ where $A = \mathrm{tridiag}(-1, 2+\gamma, -1)$, $A \in \mathbb{R}^{m\times m}$, $\gamma = c/(m+1)^2$. We find that A has eigenvalues

$$\lambda_k = \gamma + 4\sin^2\left(\frac{k\pi}{2(m+1)}\right), \quad k = 1, \ldots, m,$$

and corresponding eigenvectors

$$w_k = \begin{bmatrix} \sin\left(\frac{k\pi}{m+1}\right) \\ \sin\left(\frac{2k\pi}{m+1}\right) \\ \vdots \\ \sin\left(\frac{mk\pi}{m+1}\right) \end{bmatrix}.$$

a) Formulate the weighted Jacobi method with relaxation parameter ω for this linear system,and show that the iteration can be written in the form

$$u^{(q+1)} = G_\omega\, u^{(q)} + \frac{\omega}{2+\gamma}\, b \quad \text{where} \quad G_\omega = I - \frac{\omega}{2+\gamma}\, A,$$

and that the iteration matrix G_ω has eigenvalues

$$\mu_k = 1 - \frac{\omega}{2+\gamma}\left(\gamma + 4\sin^2\left(\frac{k\pi}{2(m+1)}\right)\right), \quad k = 1,\ldots,m,$$

and the same eigenvectors as A

Answer: Standard Jacobi results from the splitting $A = M - N = 2 + \gamma I - ((2+\gamma)I - A)$ and thus

$$(2+\gamma)\,u^{(q+1)} = ((2+\gamma)I - A)u^{(q)} + b$$

G_ω is obtained by dividing by $(2+\gamma)$ on each side. Eigenvalues and eigenvectors are found from

$$G_\omega w_k = (I - \frac{\omega}{2+\gamma}A)w_k = \left(1 - \frac{\omega}{2+\gamma}\lambda_k\right)\, w_k$$

where we insert the given λ_k.

b) The error after q iterations with weighted Jacobi on this system can be written as

$$e^{(q)} = G_\omega^q\, e^{(0)} = \sum_{k=1}^{m} \rho_k \mu_k^q w_k,$$

where

$$e^{(0)} = \sum_{k=1}^{m} \rho_k w_k$$

is the initial error $e^{(0)}$ expressed in terms of the eigenvectors w_k.

Determine the (optimal) value ω^{opt} of ω from which the best damping occurs of the upper half of the spectrum of the error, i.e. find

$$\omega^{\text{opt}} = \arg\min_{\omega} \max_{k>\frac{m+1}{2}} |\mu_k|.$$

Verify that you get back to the known $\omega^{\text{opt}} = 2/3$ when $\gamma = 0$. What happens when γ tends to infinity?

Answer: We introduce the variable $\theta = k/(m+1)$, and find that

$$\mu(\theta) = 1 - \frac{\omega}{2+\gamma}\left(\gamma + 4\sin^2\left(\frac{\pi}{2}\,\theta\right)\right)$$

$\mu(\theta)$ is a decreasing function of θ, so min and max are attained in the end points of the closed interval $\theta \in [1/2, 1]$. We conclude that the optimal ω is obtained when $\mu(1/2) = -\mu(1)$, that is

$$1 - \frac{\omega(\gamma+2)}{\gamma+2} = -\left(1 - \frac{\omega(\gamma+4)}{\gamma+2}\right)$$

which yields

$$\omega^{\text{opt}} = \frac{\gamma+2}{\gamma+3},$$

so $\gamma = 0$ leads to the well-known result $\omega^{\text{opt}} = 2/3$. When γ increases, ω^{opt} tends to the value 1.

Problem 2

a) Describe briefly the idea behind the projection methods for solving linear systems $Ax = b$. Use approximately 4-5 lines.

Answer: The idea is to define an approximation space $\mathcal{K}$ and a constraint space $\mathcal{L}$ of the same dimension, and thereafter seek, for a given x_0, a vector x such that

$$x - x_0 \in \mathcal{K}, \quad b - Ax \perp \mathcal{L}$$

Assume in the rest of this problem that $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $b \in \mathbb{R}^n$ is a given right hand side, and $x = A^{-1}b$. We let $r_k = b - Ax_k$ og $e_k = x - x_k = A^{-1}r_k$, for $k \geq 0$ and assume that x_0 is a given vector.

b) Let us use as approximation space $\mathcal{K} = \text{span}\{v\}$ and constraint space $\mathcal{L} = \mathcal{K}$. Let x_1 be the result of one step with the projection method. Show that

$$\langle Ae_1, e_1 \rangle = \langle Ae_0, e_0 \rangle - \langle r_0, v \rangle^2 / \langle Av, v \rangle.$$

Answer: The method is

$$x_1 = x_0 + \frac{\langle r_0, v \rangle}{\langle Av, v \rangle} v \quad \Rightarrow \quad e_1 = e_0 - \frac{\langle r_0, v \rangle}{\langle Av, v \rangle} v$$

Since $r_1 \perp v$

$$\langle Ae_1, e_1 \rangle = \langle r_1, e_1 \rangle = \langle r_1, e_0 \rangle = \langle Ae_0, e_0 \rangle - \frac{\langle r_0, v \rangle}{\langle Av, v \rangle} \langle Av, e_0 \rangle = \langle Ae_0, e_0 \rangle - \frac{\langle r_0, v \rangle^2}{\langle Av, v \rangle}$$

In the last step we have used that A is symmetric.

c) The method from the previous question says nothing about how the new search direction v is chosen in each iteration. Let us therefore introduce the following principle: Choose v_k such that $\langle v_k, r_k \rangle = \|r_k\|_1$, that is, let the components in v_k be 1 and -1, negative (-1) if the corresponding component in r_k is negative, and positive $(+1)$ if the r_k component is ≥ 0. Show that

$$\|e_{k+1}\|_A \leq \left(1 - \frac{1}{n\kappa(A)}\right)^{1/2} \|e_k\|_A.$$

Here $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$ while $\|w\|_A = \langle Aw, w \rangle^{1/2}$.

Answer: Rewriting the result from the previous question yields

$$\|e_{k+1}\|_A^2 = \left(1 - \frac{\langle r_k, v_k \rangle^2}{\langle Av_k, v_k \rangle \langle Ae_k, e_k \rangle}\right) \|e_k\|_A^2$$

Now use $\langle r_k, v_k\rangle = \|r_k\|_1 \geq \|r_k\|_2$, and that

$$\langle Av_k, v_k\rangle \langle Ae_k, e_k\rangle = \langle Av_k, v_k\rangle \langle r_k, A^{-1}r_k\rangle \leq \|A\|_2\|A^{-1}\|_2\|v_k\|_2^2\|r_k\|_2^2 = n\,\kappa(A)\,\|r_0\|_2^2$$

We thereby get

$$\frac{\langle r_k, v_k\rangle^2}{\langle Av_k, v_k\rangle\langle Ae_k, e_k\rangle} \geq \frac{\|r_k\|_2^2}{n\kappa(A)\|r_k\|_2^2} = \frac{1}{n\kappa(A)}$$

and as desired we obtain

$$\|e_{k+1}\|_A^2 \leq \left(1 - \frac{1}{n\kappa(A)}\right) \|e_k\|_A^2$$

Problem 3 Let the matrix A be given as

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \qquad |\varepsilon| \leq 1.$$

a) Give an estimate for the eigenvalues of A by using the Gerschgorin theorem. In particular, what can one say about the smallest eigenvalue? Make a sketch to illustrate.

Answer: All three Gerschgorin circles are disjoint, and the matrix is symmetric so the eigenvalues are real. We find that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfy

$$7 \leq \lambda_1 \leq 9, \quad 3 - \varepsilon \leq \lambda_2 \leq 5 + \varepsilon, \quad 1 - \varepsilon \leq \lambda_3 \leq 1 + \varepsilon.$$

In particular, the smallest eigenvalue is between $1 - \varepsilon$ and $1 + \varepsilon$.

b) Show for instance by using a suitable diagonal similarity transformation the sharper estimate $|\lambda_3 - 1| \leq \varepsilon^2$ for the smallest eigenvalue of A.

Answer: We compute the similarity transformation with $T = \text{diag}(1, 1, \varepsilon)$ and

$$TAT^{-1} = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & \varepsilon^2 & 1 \end{bmatrix}$$

and the third Gerschgorin disk is still disjoint from the others whenever $|\varepsilon| \leq 1$, and therefore contains an eigenvalue.

c) For $\varepsilon = 0.1$ one has found Q and R such that $A - I = QR$ where

$$Q = \begin{bmatrix} -0.9899 & 0.1413 & 0.0050 \\ -0.1414 & -0.9893 & -0.0350 \\ 0 & -0.0353 & 0.9994 \end{bmatrix}, \quad R = \begin{bmatrix} -7.0711 & -1.4142 & -0.0141 \\ 0 & -2.8302 & -0.0989 \\ 0 & 0 & -0.0035 \end{bmatrix}.$$

Find an approximation to the smallest eigenvalue of A from this.

Answer: We can easily perform *one* iteration with the shifted QR method, we just need to determine the $(3,3)$ element in $A_1 = RQ + I$ which becomes

$$1 + (-0.0035) \cdot 0.9994 = 0.9965,$$

The answer is correct in all 4 digits.

Problem 4

a) Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^T$$

b) The matrix in the previous question is a special case of a matrix $B \in \mathbb{R}^{(n+1)\times n}$ where $B_{k,k} = 1$, $B_{k+1,k} = -1$ for $k = 1, \dots, n$ and where all other elements of B are zero. Determine the singular value decomposition of B.

Answer: We must find U, V, Σ such that $B = U\Sigma V^T$. Here we find that $B^T B = \text{tridiag}(-1, 2, -1)$ for which we know the eigenvalues:

$$\sigma_k^2 = 4\sin^2\left(\frac{k\pi}{2(n+1)}\right) \quad \Rightarrow \quad \sigma_k = 2\sin\left(\frac{k\pi}{2(n+1)}\right), \quad k = 1, \dots, n.$$

The matrix V has the eigenvectors of $B^T B$ as columns, they are also known, but we need to scale them such that the Euclidian norm is 1.

$$\|w_k\|_2^2 = \sum_{j=1}^{n} \sin^2\left(\frac{jk\pi}{2(n+1)}\right) = \frac{n+1}{2}$$

So column k in V is

$$v_k = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin\left(\frac{k\pi}{n+1}\right) \\ \sin\left(\frac{2k\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nk\pi}{n+1}\right) \end{bmatrix}.$$

To find column k in U, we set $u_k = \frac{1}{\sigma_k} B v_k$. Component ℓ of u_k is therefore (at the moment we ignore the factor $\sqrt{2/(n+1)}$)

$$\frac{1}{\sigma_k}\left(\sin\left(\frac{k\ell\pi}{n+1}\right) - \sin\left(\frac{k(\ell-1)\pi}{n+1}\right)\right)$$

Here we could have called it a day, but it is in fact here all the fun begins. Letting

$$\phi = \frac{k(\ell-1/2)\pi}{n+1}, \quad \delta = \frac{k\pi/2}{n+1} \qquad \Rightarrow \sigma_k = 2\sin\delta$$

the above expression becomes

$$\frac{1}{\sigma_k}\left(\sin(\phi+\delta) - \sin(\phi-\delta)\right) = \cos\phi\frac{2\sin\delta}{\sigma_k} = \cos\phi = \cos\left(\frac{k(\ell-1/2)\pi}{n+1}\right)$$

We therefore have the following elegant expression for column k in U

$$u_k = \sqrt{\frac{2}{n+1}}\begin{bmatrix} \cos\left(\frac{k\cdot\frac{1}{2}\cdot\pi}{n+1}\right) \\ \cos\left(\frac{k\cdot\frac{3}{2}\cdot\pi}{n+1}\right) \\ \vdots \\ \cos\left(\frac{k\cdot(n+\frac{1}{2})\cdot\pi}{n+1}\right) \end{bmatrix} \in \mathbb{R}^{n+1}.$$