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Results available: 06.01.2011

EXAM IN NUMERICAL LINEAR ALGEBRA (TMA4205)

Thursday December 9, 2010
Time: 09:00–13:00

Aids: Code C,. The following printed/ hand written aids are allowed.

- Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd ed.
- Trefethen and Bau, Numerical linear algebra *or* Notes from the same book found on the course home page
- Golub and Van Loan, Matrix Computations *or* Note from the same book found on the course home page
- Own lecture notes from the course

Problem 1 A matrix $A \in \mathbb{Z}^{4\times 4}$ is being QR-factorized. After one Householder transformation using the matrix Q_1 generated by v, one has found

$$A_2 = Q_1 A = 7 \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -\frac{3}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{4}{5} & -\frac{7}{5} \end{bmatrix}, \quad v = \frac{w}{\|w\|_2}, \quad w = \begin{bmatrix} -10 \\ 0 \\ -2 \\ -6 \end{bmatrix}$$

a) Determine the original matrix A. You can use that $2\frac{w^T A_2}{w^T w} = [-1, \frac{8}{5}, -\frac{8}{5}, \frac{9}{5}]$.

Hint to check the answer: A has only integer elements.

b) Determine the upper triangular matrix R such that $A = QR$, use Householder transformations and give also the vectors v_2 and v_3 which generate Q_2 and Q_3. You are not to compute Q.

Hint to check the answer: All the elements in R are integers divisible by 7.

Problem 2 We shall apply a projection method to approximate the solution of the linear system

$$Ax = b, \qquad A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n$$

For this purpose, we use a search space $\mathcal{K}$ and a constraint space $\mathcal{L}$, both of dimension $m \leq n$. For a given initial value x_0 we seek an approximation $\tilde{x} \in x_0 + \mathcal{K}$ such that $\tilde{r} \perp \mathcal{L}$, i.e. $\tilde{r} = b - A\tilde{x}$ is orthogonal to all vectors in $\mathcal{L}$.

a) Suppose that we can write $\mathcal{L} = B\mathcal{K}$ for a nonsingular $n \times n$ matrix B. Show that if $(Ax, Bx) > 0$ for all $x \in \mathbb{R}^n$, then this method is well defined, i.e. there exists a unique $\tilde{x} \in x_0 + \mathcal{K}$ such that $\tilde{r} \perp \mathcal{L}$.

b) Assume now that B is chosen such that $C := BA^{-1}$ is symmetric positive definite. Show that the result $\tilde{x}$ will satisfy

$$\|b - A\tilde{x}\|_C = \min_{y \in x_0 + \mathcal{K}} \|b - Ay\|_C$$

where $\|\cdot\|_C$ is the vector norm on $\mathbb{R}^n$ defined as $\|v\|_C = \sqrt{v^T C v}$.

c) Let us now assume that A is symmetric so that the eigenvalues are real. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalue of A respectively. We also set $B = (1-\mu)I + \mu A$. Show that the assumptions of the previous question are satisfied such that $C = BA^{-1}$ is SPD if and only if

$$\mu < \frac{1}{1-\lambda_{\min}} \quad \text{if } \lambda_{\min} < 1 \qquad \text{and} \qquad \mu > \frac{1}{1-\lambda_{\max}} \quad \text{if } \lambda_{\max} > 1.$$

By this we mean that the first inequality can be ignored if $\lambda_{\min} \geq 1$, and the second inequality can be ignored if $\lambda_{\max} \leq 1$.

Problem 3 In this problem, we shall study in some detail the properties of splitting methods as preconditioners.

a) Let A be a nonsingular square matrix. We begin by assuming that we want to approximate the solution to the equation $A\mathrm{e} = r$ by using k iterations with a splitting method, and that we set the initial value to zero, i.e. $e^{(0)} = 0$. Assume first a general splitting $A = D - N$, D invertible, and an iteration of the form

$$\mathrm{e}^{(k+1)} = G\mathrm{e}^{(k)} + \bar{r}, \qquad G = D^{-1}N, \quad \bar{r} = D^{-1}r$$

Show that one has $\mathrm{e}^{(k)} = (I - G)^{-1}(I - G^k)\bar{r}$, and if the corresponding preconditioned system is $\tilde{A}x = M^{-1}Ax = M^{-1}b$ then one has

$$\tilde{A} = M^{-1}A = (I - G)^{-1}(I - G^k)(I - G).$$

b) Assume in the rest of this problem that A is symmetric positive definite (SPD) of the form $A = \alpha I - N$, $N^T = N$, $\alpha > \frac{1}{2}\lambda_{\max}$, where $\lambda_{\max} = \rho(A)$ is the largest eigenvalue of A. Let $D = \alpha I$. Show that the preconditioner M from the question above then will also be SPD

c) Suppose as before that A is SPD, and that the smallest and largest eigenvalue of A are $\lambda_{\min}$ and $\lambda_{\max}$ respectively. We choose the splitting parameter $\alpha = \frac{1}{2}(\lambda_{\min} + \lambda_{\max})$ such that the preconditioner is SPD. Suppose that we use k iterations of the splitting method where k is an odd integer. Show that under these circumstances one has

$$\kappa_2(\tilde{A}) = \frac{1 + \left(\frac{\kappa - 1}{\kappa + 1}\right)^k}{1 - \left(\frac{\kappa - 1}{\kappa + 1}\right)^k}$$

where $\kappa = \kappa_2(A)$ is the condition number of A.

Comment on the result.

Problem 4

a) Show that the Frobenius norm of an $n \times n$ matrix can be written as

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2},$$

where $\sigma_1, \ldots, \sigma_n$ are the singular values of A.

b) Suppose that A is a 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Find from this the largest possible lower bound for $\kappa_2(A) = \|A\|_2 \, \|A^{-1}\|_2$.

Appendix. Some useful formulas

1. For all $n \times n$ matrices C with elements c_{ij} and eigenvalues λ_i one has

$$\mathrm{Tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \lambda_i$$

2. The condition number of a matrix A is given by the formula $\kappa(A) = \|A\| \|A^{-1}\|$. In particular, using the p-norm, one writes $\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$