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EXAM IN NUMERICAL LINEAR ALGEBRA (TMA4205)

Thursday December 9, 2010
Time: 09:00–13:00

Aids: Code C,. The following printed/ hand written aids are allowed.

- Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd ed.
- Trefethen and Bau, Numerical linear algebra *or* Notes from the same book found on the course home page
- Golub and Van Loan, Matrix Computations *or* Note from the same book found on the course home page
- Own lecture notes from the course

Problem 1 A matrix $A \in \mathbb{Z}^{4\times 4}$ is being QR-factorized. After one Householder transformation using the matrix Q_1 generated by v, one has found

$$A_2 = Q_1 A = 7 \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -\frac{3}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{4}{5} & -\frac{7}{5} \end{bmatrix}, \quad v = \frac{w}{\|w\|_2}, \quad w = \begin{bmatrix} -10 \\ 0 \\ -2 \\ -6 \end{bmatrix}$$

a) Determine the original matrix A. You can use that $2\dfrac{w^T A_2}{w^T w} = [-1, \frac{8}{5}, -\frac{8}{5}, \frac{9}{5}]$.
Hint to check the answer: A has only integer elements.

Answer: Use the fact that Householder transformations are their own inverse, so $A = Q_1^{-1}A_2 = Q_1 A_2$. We need to find $(I - 2vv^T)A_2$

$$= A_2 - 2w\frac{w^T A_2}{w^T w} = 7 \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -\frac{3}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{4}{5} & -\frac{7}{5} \end{bmatrix} - \begin{bmatrix} -10 \\ 0 \\ -2 \\ -6 \end{bmatrix} \cdot [-1, \frac{8}{5}, -\frac{8}{5}, \frac{9}{5}] = \begin{bmatrix} -3 & 9 & -9 & 11 \\ 0 & 0 & 7 & -7 \\ -2 & -1 & 1 & 5 \\ -6 & 4 & -4 & 1 \end{bmatrix}$$

b) Determine the upper triangular matrix R such that $A = QR$, use Householder transformations and give also the vectors v_2 and v_3 which generate Q_2 and Q_3. You are not to compute Q.

Hint to check the answer: All the elements in R are integers divisible by 7.

Answer: We begin by setting $x = A_2(2:4, 2) = 7 \cdot [0, -3/5, -4/5]^T$ and use the formula $v = w/\|w\|_2$ where $w = x + \text{sign}(x_1)\|x\|$. Since $\|x\|_2 = 7$ we get $w = 7 \cdot [1, -3/5, -4/5]^T := 7 \cdot \tilde{w}$, such that $\tilde{w}^T\tilde{w} = 2$. Let $\tilde{A}_2 = A_2(2:4, 2:4)$, we just change this part of A_2 to find A_3. One has $\tilde{Q}_2\tilde{A}_2 = (I - 2vv^T)\tilde{A}_2$

$$\tilde{A}_3 = \tilde{A}_2 - 2\frac{ww^T}{w^T w}\tilde{A}_2 = \tilde{A}_2 - 2\tilde{w}\frac{\tilde{w}^T\tilde{A}_2}{\tilde{w}^T\tilde{w}} = \tilde{A}_2 - \tilde{w}\tilde{w}^T\tilde{A}_2 = \tilde{A}_2 - 7\tilde{w} \cdot [1, 0, 0]$$

Thus

$$\tilde{A}_3 = 7 \cdot \begin{bmatrix} 0 & 1 & -1 \\ -\frac{3}{5} & \frac{3}{5} & \frac{1}{5} \\ -\frac{4}{5} & \frac{4}{5} & -\frac{7}{5} \end{bmatrix} - 7 \cdot \begin{bmatrix} 1 \\ -\frac{3}{5} \\ -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = 7 \cdot \begin{bmatrix} -1 & 1 & -1 \\ 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{4}{5} & -\frac{7}{5} \end{bmatrix}.$$

One can use for the Householder transformation, $v_2 = \sqrt{1/2}\,[1, -3/5, -4/5]^T$.

We proceed in the same way and define $\tilde{\tilde{A}}_3 = \tilde{A}_3(2:3, 2:3)$. Set $x = 7 \cdot [3/5, 4/5]^T$, and find $\|x\|_2 = 7$, and thus $w = 7 \cdot [3/5 + 1, 4/5]^T = 28 \cdot [2/5, 1/5]^T = 28 \cdot \tilde{w}$, where $\tilde{w}^T\tilde{w} = 1/5$. The same type of computation as above yields

$$\tilde{A}_4 = 7 \cdot \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

We substitute the matrices back into each other to obtain

$$R = 7 \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

One may define $v_3 = \sqrt{1/5} \cdot [2, 1]^T$.

Problem 2 We shall apply a projection method to approximate the solution of the linear system

$$Ax = b, \qquad A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n$$

For this purpose, we use a search space $\mathcal{K}$ and a constraint space $\mathcal{L}$, both of dimension $m \le n$. For a given initial value x_0 we seek an approximation $\tilde{x} \in x_0 + \mathcal{K}$ such that $\tilde{r} \perp \mathcal{L}$, i.e. $\tilde{r} = b - A\tilde{x}$ is orthogonal to all vectors in $\mathcal{L}$.

a) Suppose that we can write $\mathcal{L} = B\mathcal{K}$ for a nonsingular $n \times n$ matrix B. Show that if $(Ax, Bx) > 0$ for all $x \in \mathbb{R}^n$, then this method is well defined, i.e. there exists a unique $\tilde{x} \in x_0 + \mathcal{K}$ such that $\tilde{r} \perp \mathcal{L}$.

Answer: We introduce a basis for $\mathcal{K}$ as the columns of the $n \times m$ matrix, $V_m = [v_1|\cdots|v_m] \in \mathbb{R}^{n\times n}$. The vectors $w_j = Bv_j, j = 1,\ldots,m$ then form a linearly independent set in $\mathcal{L}$, which we can take as a basis, we set $W_m = [w_1|\cdots|w_m] \in \mathbb{R}^{n\times n}$. Any vector $\tilde{x}$ in $x_0 + \mathcal{K}$ can be written in the form $\tilde{x} = x_0 + V_m y$, $y \in \mathbb{R}^m$. We have

$$\tilde{r} = b - A\tilde{x} = b - Ax_0 - AV_m y = r_0 - AV_m y.$$

We require that $\tilde{r} \perp \mathcal{L}$ which is equivalent to $w_j \perp \tilde{r},\ r = 1,\ldots,m$ or $W_m^T \tilde{r} = 0$. This yields

$$(BV_m)^T(r_0 - AV_m y) = 0 \quad \Leftrightarrow \quad (V_m^T B^T A V_m)y = V_m^T B^T r_0$$

A unique solution exists if and only if $P = V_m^T B^T A V_m \in \mathbb{R}^{m\times m}$ is nonsingular. A sufficient condition for this is that P is positive definite, i.e. $z^T P z > 0\ \forall z \in \mathbb{R}^m$. We show that it is exactly that under the assumption $(Ax, Bx) > 0$. We have

$$z^T P z = z^T V_m^T B^T A V_m z = (Bx)^T(Ax) = (Ax, Bx) > 0, \qquad x = V_m z.$$

b) Assume now that B is chosen such that $C := BA^{-1}$ is symmetric positive definite. Show that the result $\tilde{x}$ will satisfy

$$\|b - A\tilde{x}\|_C = \min_{y\in x_0+\mathcal{K}} \|b - Ay\|_C$$

where $\|\cdot\|_C$ is the vector norm on $\mathbb{R}^n$ defined as $\|v\|_C = \sqrt{v^T C v}$.

Answer: First a minor comment. That such a B exists we can see for instance by taking the candidates $B = I$ or $B = A$ where we have assumed that A is nonsingular. Let us perturb the given candidate, by setting $y = \tilde{x} + \delta,\ \delta \in \mathcal{K}$ and then study $\|b - Ay\|_C$

$$\|b - Ay\|_C^2 = \|b - A\tilde{x} - A\delta\|_C^2 = (\tilde{r} - A\delta)^T C(\tilde{r} - A\delta) = \|\tilde{r}\|_C^2 + \|A\delta\|_C^2 - 2\tilde{r}^T BA^{-1} A\delta.$$

We notice that the last term vanishes because $\tilde{r} \perp B\delta$ siden $B\delta \in \mathcal{L}$. Therefore

$$\|b - Ay\|_C \ge \|\tilde{r}\|_C$$

and equality requires $A\delta = 0$ which again calls for $\delta = 0$ i.e. $y = \tilde{x}$, the minimum is unique.

c) Let us now assume that A is symmetric so that the eigenvalues are real. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalue of A respectively. We also set $B = (1-\mu)I + \mu A$. Show that the assumptions of the previous question are satisfied such that $C = BA^{-1}$ is SPD if and only if

$$\mu < \frac{1}{1-\lambda_{\min}} \quad \text{if } \lambda_{\min} < 1 \qquad \text{and} \qquad \mu > \frac{1}{1-\lambda_{\max}} \quad \text{if } \lambda_{\max} > 1.$$

By this we mean that the first inequality can be ignored if $\lambda_{\min} \geq 1$, and the second inequality can be ignored if $\lambda_{\max} \leq 1$.

Answer: We now find that

$$C = BA^{-1} = ((1-\mu)I + \mu A)A^{-1} = (1-\mu)A^{-1} + \mu I$$

C is obviously symmetric if A is symmetric. Considering the eigenvalues of C we have

$$\lambda(C) = \frac{1-\mu}{\lambda(A)} + \mu$$

and we need only demand that they are positive. For those eigenvalues of A which are larger than 1, it is the biggest of them, namely $\lambda_{\max}$ which determines the most restrictive condition. For those less than 1, it is $\lambda_{\min}$ which determines the condition.

Problem 3 In this problem, we shall study in some detail the properties of splitting methods as preconditioners.

a) Let A be a nonsingular square matrix. We begin by assuming that we want to approximate the solution to the equation $A\mathrm{e} = r$ by using k iterations with a splitting method, and that we set the initial value to zero, i.e. $e^{(0)} = 0$. Assume first a general splitting $A = D - N$, D invertible, and an iteration of the form

$$\mathrm{e}^{(k+1)} = G\mathrm{e}^{(k)} + \bar{r}, \qquad G = D^{-1}N, \quad \bar{r} = D^{-1}r$$

Show that one has $\mathrm{e}^{(k)} = (I-G)^{-1}(I-G^k)\bar{r}$, and if the corresponding preconditioned system is $\tilde{A}x = M^{-1}Ax = M^{-1}b$ then one has

$$\tilde{A} = M^{-1}A = (I-G)^{-1}(I-G^k)(I-G).$$

Answer: We can prove this by induction. Tha $\mathrm{e}^{(1)} = \bar{r}$ can be seen directly from the iteration formula with $k=0$, since $\mathrm{e}^{(0)} = 0$. If the formula is correct up to $k-1$ we can find $\mathrm{e}^{(k)}$ by

$$\mathrm{e}^{(k)} = G\mathrm{e}^{(k-1)} + \bar{r} = (I + G(I-G)^{-1}(I-G^{k-1}))\bar{r} = (I-G)^{-1}(I-G+G(I-G^{k-1}))\bar{r} = (I-G)^{-1}(I-G^k)\bar{r}$$

To find the expression for $\tilde{A}$ we need to substitute $\bar{r} = D^{-1}r$ and use that $A = D(I-G)$.

$$\tilde{A} = M^{-1} \cdot A = (I-G)^{-1}(I-G^k)D^{-1} \cdot (D-N) = (I-G)^{-1}(I-G^k)(I-G)$$

b) Assume in the rest of this problem that A is symmetric positive definite (SPD) of the form $A = \alpha I - N$, $N^T = N$, $\alpha > \frac{1}{2}\lambda_{\max}$, where $\lambda_{\max} = \rho(A)$ is the largest eigenvalue of A. Let $D = \alpha I$. Show that the preconditioner M from the question above then will also be SPD

Answer: We have $M^{-1} = (I-G)^{-1}(I-G^k)D^{-1} = \frac{1}{\alpha}(I-G)^{-1}(I-G^k)$ where $G = \frac{1}{\alpha}N$ is symmetric. Therefore both $(I-G)^{-1}$ and $I - G^k$ are symmetric and we have $M^{-T} = (I-G^k)(I-G)^{-1} = (I-G)^{-1}(I-G^k) = M^{-1}$, so M^{-1} (and thus M) is symmetric.

Now $\lambda(G) = 1 - \dfrac{\lambda(A)}{\alpha}$ such that

$$\lambda(M^{-1}) = \frac{1}{\alpha}(1-\lambda(G))^{-1}(1-\lambda(G)^k) = \frac{1}{\alpha}\frac{\alpha}{\lambda(A)}\left(1 - \left(1 - \frac{\lambda(A)}{\alpha}\right)^k\right)$$

So $\lambda(M^{-1}) > 0$ if $1 - \left(1 - \frac{\lambda(A)}{\alpha}\right)^k > 0$. Since $\alpha > \frac{1}{2}\lambda_{\max}$ we must have $-1 < 1 - \frac{\lambda(A)}{\alpha} < 1$ for all eigenvalues of A, and therefore $-1 < (1 - \frac{\lambda(A)}{\alpha})^k < 1$. It follows that $1 - \left(1 - \frac{\lambda(A)}{\alpha}\right)^k > 0$ and therefore $\lambda(M^{-1}) > 0$, so M^{-1} (and thus M) is SPD.

c) Suppose as before that A is SPD, and that the smallest and largest eigenvalue of A are $\lambda_{\min}$ and $\lambda_{\max}$ respectively. We choose the splitting parameter $\alpha = \frac{1}{2}(\lambda_{\min} + \lambda_{\max})$ such that the preconditioner is SPD. Suppose that we use k iterations of the splitting method where k is an odd integer. Show that under these circumstances one has

$$\kappa_2(\tilde{A}) = \frac{1 + \left(\frac{\kappa-1}{\kappa+1}\right)^k}{1 - \left(\frac{\kappa-1}{\kappa+1}\right)^k}$$

where $\kappa = \kappa_2(A)$ is the condition number of A.

Comment on the result.

Answer: The condition number of A is given as $\kappa = \kappa_2(A) = \frac{\lambda_{\max}}{\lambda_{\min}}$. Let $\tilde{\lambda} \in \sigma(\tilde{A})$. Since $\tilde{A} = (I-G)^{-1}(I-G^k)(I-G)$ then $\tilde{\lambda} \in \sigma(I - G^k)$. Because $\alpha = \frac{1}{2}(\lambda_{\min} + \lambda_{\max})$ we have

$$\tilde{\lambda} = 1 - \left(1 - \frac{2\lambda}{\lambda_{\min} + \lambda_{\max}}\right)^k, \quad \lambda \in \sigma(A).$$

One can see that this expression is monotonically increasing in λ for instance by differentiating

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\lambda} = \frac{2k}{\lambda_{\min} + \lambda_{\max}}\left(1 - \frac{2\lambda}{\lambda_{\min} + \lambda_{\max}}\right)^{k-1}$$

Since k is odd, $k-1$ is even, and the expression is non-negative. Therefore the minimum is attained at $\lambda = \lambda_{\min}$ and the maximum at $\lambda = \lambda_{\max}$. We compute

$$\kappa_2(\tilde{A}) = \frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}} = \frac{1 - \left(1 - \frac{2\lambda_{\max}}{\lambda_{\min}+\lambda_{\max}}\right)^k}{1 - \left(1 - \frac{2\lambda_{\min}}{\lambda_{\min}+\lambda_{\max}}\right)^k} = \frac{1 + \left(\frac{\kappa-1}{\kappa+1}\right)^k}{1 - \left(\frac{\kappa-1}{\kappa+1}\right)^k}$$

Problem 4

a) Show that the Frobenius norm of an $n \times n$ matrix can be written as

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2},$$

where $\sigma_1, \ldots, \sigma_n$ are the singular values of A.

Answer: The Frobenius norm of A can be written in the form

$$\|A\|_F = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2} = \sqrt{\mathrm{Tr}(A^T A)}$$

Then one ay just use the given information that the trace of a matrix equals the sum of its eigenvalues and that the square of the singular values of A are the eigenvalues of $A^T A$, the result then follows.

b) Suppose that A is a 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Find from this the largest possible lower bound for $\kappa_2(A) = \|A\|_2 \, \|A^{-1}\|_2$.

Answer: Here we use the answer from the previous question. Note that $\|A\|_2 = \sigma_1$ i.e. the largest singular value, whereas $\|A^{-1}\|_2 = 1/\sigma_n$ i.e. the inverse of the smallest singular value, here $n = 202$. One finds

$$201 = 101^2 - 100^2 = \|A\|_F^2 - \|A\|_2^2 = \sum_{k=2}^{202} \sigma_k^2 \geq 201 \cdot \sigma_{202}^2$$

such that $\sigma_{202} \leq 1$. Therefore

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_{202}} \geq \frac{100}{1} = 100.$$

Appendix. Some useful formulas

1. For all $n \times n$ matrices C with elements c_{ij} and eigenvalues λ_i one has

$$\mathrm{Tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \lambda_i$$

2. The condition number of a matrix A is given by the formula $\kappa(A) = \|A\| \|A^{-1}\|$. In particular, using the p-norm, one writes $\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$