NTNU

Department of Mathematical Sciences

Examination paper for **TMA4205 Numerical Linear Algebra**

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Permitted examination support material: C: Specified, written and handwritten examination support materials are permitted. A specified, simple calculator is permitted (either Citizen SR-270X or Hewlett Packard HP30S). The permitted examination support materials are:

- Y. Saad: Iterative Methods for Sparse Linear Systems. 2nd ed. SIAM, 2003 (book or printout)
- L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997 (book or photocopy)
- G. Golub and C. Van Loan: Matrix Computations. 3rd ed. The Johns Hopkins University Press, 1996 (book or photocopy)
- E. Rønquist: Note on The Poisson problem in $\mathbb{R}^2$: diagonalization methods (printout)
- K. Rottmann: Matematisk formelsamling
- Your own lecture notes from the course (handwritten)

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Problem 1 Let A be the $n \times n$ Toeplitz matrix (i.e. a matrix where the elements of each diagonal are equal to each other) given by

$$A = d \begin{bmatrix} 1 & 1/3 & 1/9 & \cdots & 1/3^{n-1} \\ 1/3 & 1 & 1/3 & \cdots & 1/3^{n-2} \\ 1/9 & 1/3 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1/3 \\ 1/3^{n-1} & 1/3^{n-2} & \cdots & 1/3 & 1 \end{bmatrix},$$

where d is a nonzero, real scalar.

a) Is A normal?

b) Are the eigenvalues of A real or complex?

c) Is A positive definite?

d) Is A nonsingular?

We now apply Jacobi iteration to solve the system of linear equations $Ax = b$, where A is given above.

e) Will the Jacobi iteration converge for any initial vector?

f) Let L be the lower-triangular part of A (including the diagonal). Find L^{-1}.

Hint: You may use the fact that the inverse of a lower-triangular Toeplitz matrix is also a lower-triangular Toeplitz matrix.

If you did not find L^{-1} in **f)**, you may from now on use L^{-1} equal to the Toeplitz matrix with $1/(3d)$ on the diagonal and $-1/(9d)$ on the subdiagonal, and zero elsewhere. Note that this L^{-1} is not the correct answer to **f)**.

g) Instead of using Jacobi iteration, we will now use Gauss–Seidel iteration. What is the spectral radius of the iteration matrix used in the Gauss–Seidel iteration?

Problem 2 Consider the 2D Helmholtz equation

$$-\nabla^2 u - \alpha u = f \quad \text{in} \quad \Omega = (0,1)\times(0,1),$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where α is a positive constant, and $f\colon \Omega \to \mathbb{R}$. Using centered finite differences and based on the diagonalization method for the 2D Poisson equation[1], construct a diagonalization method for solving the 2D Helmholtz equation.

Problem 3

a) In a few sentences, explain what a Krylov subspace is, and what the Arnoldi process does.

Let $A = I + B$, where I is the identity matrix and B is a skew-symmetric matrix.

b) Consider the Arnoldi process for A. Show that the resulting Hessenberg matrix will have the tridiagonal form

$$H_m = \begin{bmatrix} 1 & -\beta_2 & & \\ \beta_2 & 1 & \ddots & \\ & \ddots & \ddots & -\beta_m \\ & & \beta_m & 1 \end{bmatrix}.$$

c) Show that by exploiting the structure of our matrix $A = I + B$, we can simplify the Arnoldi MGS (modified Gram–Schmidt) process to:

$r_0 = b - Ax_0$, $\beta_1 = \|r_0\|_2$, $v_1 = r_0/\beta_1$, $v_0 = 0$
for $j = 1, \dots, m$ **do**
  $w_j = Bv_j + \beta_j v_{j-1}$
  $\beta_{j+1} = \|w_j\|_2$
  $v_{j+1} = w_j/\beta_{j+1}$
end for

Hint: This is similar to the Lanczos process.

d) H_m may be LU-factorized into

$$H_m = L_m U_m = \begin{bmatrix} 1 & & & \\ \lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_m & 1 \end{bmatrix} \begin{bmatrix} \eta_1 & -\beta_2 & & \\ & \ddots & \ddots & \\ & & \eta_{m-1} & -\beta_m \\ & & & \eta_m \end{bmatrix}.$$

[1] See the note by E. Rønquist.

Use this LU-factorization to find an algorithm analogous to the direct Lanczos (D-Lanczos) algorithm, but applied to our non-symmetric matrix A.

Problem 4

a) Find a singular value decomposition (SVD) of

$$M = \begin{bmatrix} 48 & 36 & 20 \\ 36 & 27 & 15 \\ 20 & 15 & 75 \end{bmatrix}.$$

Hint: The singular values are integers, and the largest singular value is double the middle singular value.

b) How are the singular values and eigenvalues of M related?

c) Use the SVD to find the rank of M.

d) Find an approximation $\tilde{M} \approx M$ so that $\operatorname{rank} \tilde{M} = 1$, and $\|M - \tilde{M}\|_{\mathrm{F}}$ is minimal. Here, $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm.