Problem 1

a) $B = \begin{pmatrix} 0 & \mathcal{A} \\ \mathcal{A} & 0 \end{pmatrix}$ $\qquad \det(B-\lambda I) = \lambda^2 - \mathcal{A}^2$

$$\lambda_\pm = \pm \mathcal{A}$$

$$v_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) First of all: B is Hermitian $\Rightarrow$

$$B^H = Q R^H Q^H = B = Q R Q^H$$

$\Rightarrow R^H = R \Rightarrow R$ is diagonal, with real diagonal.

Therefore Q – eigenvectors

R – diagonal with eigenvalues on the diagonal

Hint from a): eigenvalues of B are $\pm \Sigma$

and eigenvectors related to U, V

Alternatively one can look at $BB^H = B^2 = \begin{pmatrix} A^H A & 0 \\ 0 & AA^H \end{pmatrix}$

to realize that the eigenvalues of B must be related to Σ and the eigenvectors to U, V.

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \begin{pmatrix} Q_{11}^H & Q_{21}^H \\ Q_{12}^H & Q_{22}^H \end{pmatrix}$$

$$= \begin{pmatrix} Q_{11}\Sigma & -Q_{12}\Sigma \\ Q_{21}\Sigma & -Q_{22}\Sigma \end{pmatrix} \begin{pmatrix} Q_{11}^H & Q_{21}^H \\ Q_{12}^H & Q_{22}^H \end{pmatrix}$$

$$= \begin{pmatrix} Q_{11}\Sigma Q_{11}^H - Q_{12}\Sigma Q_{12}^H & Q_{11}\Sigma Q_{21}^H - Q_{12}\Sigma Q_{22}^H \\ Q_{21}\Sigma Q_{11}^H - Q_{22}\Sigma Q_{12}^H & Q_{21}\Sigma Q_{21}^H - Q_{22}\Sigma Q_{22}^H \end{pmatrix} \stackrel{!}{=}$$

$$= \begin{pmatrix} 0 & A^H \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & V\Sigma U^H \\ U\Sigma V^H & 0 \end{pmatrix}$$

The equation above is e.g. satisfied when

$Q_{21} = \frac{1}{\sqrt{2}} U \qquad Q_{22} = -\frac{1}{\sqrt{2}} U$

$Q_{11} = \frac{1}{\sqrt{2}} V \qquad Q_{12} = \frac{1}{\sqrt{2}} V$

Thus : $\quad R = \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}; \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} V & V \\ U & -U \end{pmatrix}$

Problem 2

a) $X_{new} = x + \delta v$

$r_{new} = r - \delta A v \quad \perp Av$

$$\delta = \frac{r^T A v}{\|Av\|_2^2}$$

$$\|r_{new}\|_2^2 = \|r\|_2^2 + \frac{(r^T A v)^2}{\|Av\|_2^4} \cdot \|Av\|_2^2$$

$$- 2 \frac{r^T A v}{\|Av\|_2^2} \cdot r^T A v$$

$$= \|r\|_2^2 - \frac{(r^T A v)^2}{\|Av\|_2^2}$$

b) Pick $0 \neq v \perp A^T r$, which is always possible in dimension $n > 1$

c) $r^T A v = \|A^T r\|_1 \geq \|A^T r\|_2 \geq \dfrac{\|r\|_2}{\|A^{-T}\|_2} = \dfrac{\|r\|_2}{\|A^{-1}\|_2}$

because $\sigma_1(A) = \sigma_1(A^T)$

$\|Av\|_2 \leq \|A\|_2 \cdot \|v\|_2 \leq \sqrt{n}\, \|A\|_2$

as each component of v, $|v_i| \leq 1$.

$$\Rightarrow \|r_{new}\|^2 \leq \|r\|_2^2 - \frac{\|r\|_2^2}{\left(\sqrt{n}\, \|A\|_2 \|A^{-1}\|_2\right)^2}$$

$\boxed{}$

Problem 3

a) T_m is tri-diagonal $= \begin{pmatrix} \alpha_1 & \beta_2 & & 0 \\ \beta_2 & \ddots & \ddots & \\ & \ddots & & \beta_m \\ 0 & & \beta_m & \alpha_m \end{pmatrix}$

We apply Givens rotation G_{12} on the left, which operates on the first two rows only & removes β_2

$$G_{12} T_m = \begin{pmatrix} * & * & * & 0 & \cdots \\ 0 & * & * & 0 & \\ 0 & \beta_3 & \alpha_3 & \beta_4 & \cdots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Application of this rotation introduces a non-zero element on the second super-diagonal (results from rotating a vector $\begin{pmatrix} 0 \\ \beta_3 \end{pmatrix}$ in the 3^{rd} column). Only 3 columns need to be rotated (T_m is tri-diagonal).

We then proceed to eliminating β_3 from position $(3,2)$ by applying a Givens rotation $G_{3,2}$ (which also introduces only 1 non-zero on the second superdiagonal and only needs to rotate 3 vectors).

All in all $m-1$ rotations are needed to eliminate the sub-diagonal. Each rotation modifies only 3 2-dim vectors & introduces 1 non-zero on the 2^{nd} superdiagonal $\Rightarrow$

$$R_m = \underbrace{G_{m,m-1} \cdot \ldots \cdot G_{12}}_{=: Q_m^T} T_m$$

– recall, product of orth. matrices is orthogonal.

b) Since the projection algorithm is equivalent to FOM, we get

$$x_m = x_0 + V_m T_m^{-1} \|r_0\| e_1$$

(formula 6.17 in Saad) or re-derive!

$$T_m = T_m^T \quad (A \text{ is symmetric})$$

$$\Rightarrow T_m^{-1} = T_m^{-T} = (Q_m R_m)^{-T} =$$

$$= Q_m^{-T} R_m^{-T} = Q_m R_m^{-T}, \text{ because } Q_m^{-1} = Q_m^T.$$

c)

$$T_m = \left(\begin{array}{ccc|c} & T_{m-1} & & \begin{matrix}0\\0\\0\\\beta_m\end{matrix} \\ \hline 0 & 0\;0\;0 & \beta_m & \alpha_m \end{array}\right) = \left(\begin{array}{c|c} Q_{m-1} R_{m-1} & \begin{matrix}0\\0\\\beta_m\end{matrix} \\ \hline 0\;0\;0\;\ldots\;\beta_m & \alpha_m \end{array}\right)$$

$$= \left(\begin{array}{c|c} Q_{m-1} & \begin{matrix}0\\\vdots\\0\end{matrix} \\ \hline 0\;0\;\ldots\;0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} R_{m-1} & Q_{m-1}^T \begin{pmatrix}0\\0\\0\\\beta_m\end{pmatrix} \\ \hline 0\;\ldots\;0\;\beta_m & \alpha_m \end{array}\right)$$

$$= \begin{pmatrix} Q_{m-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{array}{c|c} R_{m-1} & G_{m-1,m-2} \begin{pmatrix}0\\0\\\beta_m\end{pmatrix} \\ \hline 0\;\ldots\;0\;\beta_m & \alpha_m \end{array}\right) \qquad (*)$$

$$Q_{m-1}^T \begin{pmatrix}0\\\vdots\\0\\\beta_m\end{pmatrix} = G_{m-2,m-1} \begin{pmatrix}0\\\vdots\\0\\\beta_m\end{pmatrix}$$ because all other rotations would operate on the components of the vector, which are zero

It remains to eliminate β_m from the sub-diagonal of the second matrix on the right hand side of $(*)$;
this also affects the bottom right element of R_{m-1}!
The rotation needs to be applied to two vectors only!
(last two columns)

Thus

$$T_m = \begin{pmatrix} Q_{m-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot G_{m-1,m}^T \; G_{m-1,m} \left(\begin{array}{c|c} R_{m-1} & G_{m-2,m-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_m \end{pmatrix} \\ 0 \cdots 0 \;\; \beta_m & \alpha_m \end{array} \right)$$

$$\boxed{= \underbrace{\begin{pmatrix} Q_{m-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot G_{m-1,m}^T}_{=:\, Q_m} \underbrace{\left(\begin{array}{c|c} \tilde{R}_{m-1} & r_m \\ 0 \cdots 0 & \end{array} \right)}_{=:\, R_m}}$$

Note: tri-diagonal structure of R_m is maintained ($\tilde{R}_{m-1}$ is 3-diagonal;

$G_{m-2,m-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_m \end{pmatrix}$ has only two last components,

which are not zero)

Problem 4

a) $x^{(0)} = 0$

$$x^{(1)} = A_1^{-1}(A_2 x^{(0)} + b) = A_1^{-1} b$$

$$x^{(2)} = A_1^{-1}(A_2 A_1^{-1} b + b) =$$

$$= A_1^{-1}(A_2 A_1^{-1} + I)\, b$$

$$x^{(3)} = A_1^{-1}(A_2 \underbrace{A_1^{-1}(A_2 A_1^{-1} + I) b}_{x^{(2)}} + b)$$

$$= A_1^{-1}(A_2 A_1^{-1}(A_2 A_1^{-1} + I) + I)\, b$$

$$= A_1^{-1}\left[(A_2 A_1^{-1})^2 + A_2 A_1^{-1} + I\right] b$$

After ν iterations:

$$\boxed{x^{(\nu)} = \underbrace{A_1^{-1}\left[\sum_{i=0}^{\nu-1} (A_2 A_1^{-1})^i\right]}_{=:M^{-1}\,!} b}$$

$$x^{(\nu+1)} = A_1^{-1}\left[A_2 A_1^{-1} \sum_{i=0}^{\nu-1} (A_2 A_1^{-1})^i + I\right] b$$

$$= A_1^{-1}\left[\sum_{i=0}^{\nu} (A_2 A_1^{-1})^i\right] b$$

$$\boxed{A M^{-1} = A A_1^{-1} \sum_{i=0}^{\nu-1} (A_2 A_1^{-1})^i}$$

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preconditioner

b) The number of iterations ν may vary from one Arnoldi iteration to another.

Thus $M^{-1} = AA^{-1} \sum_{i=1}^{\nu-1} (A_2 A_1)^{-1}$ will not be constant any longer!

FGMRES (flexible GMRES) has been specifically designed for such preconditioning strategies.