Department of Mathematical Sciences

Examination paper for **TMA4205 Numerical Linear Algebra**

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Permitted examination support material: C: Specified, written and handwritten examination support materials are permitted. A specified, simple calculator is permitted. The permitted examination support materials are:

- Y. Saad: Iterative Methods for Sparse Linear Systems. 2nd ed. SIAM, 2003 (book or printout)
- L. N. Trefethen and D. Bau: Numerical Linear Algebra, SIAM, 1997 (book or photocopy)
- G. Golub and C. Van Loan: Matrix Computations. 3rd ed. The Johns Hopkins University Press, 1996 (book or photocopy)
- J. W. Demmel: Applied Numerical Linear Algebra, SIAM, 1997 (book or printout)
- E. Rønquist: Note on The Poisson problem in $\mathbb{R}^2$: diagonalization methods (printout)
- K. Rottmann: Matematisk formelsamling
- Your own lecture notes from the course (handwritten)

Language: English
Number of pages: 3
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Problem 1

a) A 1×1 matrix $A = (\alpha)$, $\alpha > 0$ trivially admits an SVD decomposition $A = U\Sigma V^H$ with $\Sigma = (\alpha)$ and $U = V = (1)$. Consider a related Hermitian matrix

$$B = \begin{pmatrix} 0 & A^H \\ A & 0 \end{pmatrix}, \tag{1}$$

and compute its eigenvalues and eigenvectors.

b) Let now $A \in \mathbb{C}^{n\times n}$ be an arbitrary square matrix with a singular value decomposition $A = U\Sigma V^H$. Let B be defined by (1), and let $B = QRQ^H$ be its Schur canonical form. Using **a)** as an insight, show how Q and R can be expressed in terms of U, Σ, and V.

Problem 2 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric non-singular real matrix, and $b, x \in \mathbb{R}^n$ be some given vectors. Let $r = b - Ax$.

a) Let $v \in \mathbb{R}^n$ be an arbitrary non-zero vector, and consider a one-dimensional projection method with $\mathcal{K} = \operatorname{span}\langle v\rangle$ and $\mathcal{L} = A\mathcal{K}$.

Let x_{new} be the result of one step of this projection method starting from x, and let r_{new} be the associated residual. Show that

$$\|r_{\text{new}}\|_2^2 = \|r\|_2^2 - \left[\frac{r^T Av}{\|Av\|_2}\right]^2.$$

b) The previous inequality shows that $\|r_{\text{new}}\|_2 \leq \|r\|_2$. Show that for every $n > 1$, A, b, and x it is possible to select $v \neq 0$ such that $\|r_{\text{new}}\|_2 = \|r\|_2$. That is, this method does not necessarily converge.

c) Let us now select components of vector v in the previous part as follows: $v_i = \operatorname{sign}([A^T r]_i)$, where $\operatorname{sign} : \mathbb{R} \to \{-1, 0, 1\}$ is the sign function (we assume $\operatorname{sign}(0) = 0$). Show that with this choice of v we have

$$\|r_{\text{new}}\|_2^2 \leq \left[1 - \frac{1}{n\kappa_2^2(A)}\right]\|r\|_2^2,$$

where $\kappa_2(A)$ is the spectral condition number of A. You may find the following inequalities, which are valid for any $z \in \mathbb{R}^n$, helpful: $\|z\|_2 \leq \|z\|_1$, and $\|Az\|_2 \geq \|z\|_2/\|A^{-1}\|_2$.

Problem 3 Consider a *symmetric*, possibly indefinite real matrix $A \in \mathbb{R}^{n\times n}$. Let $T_m \in \mathbb{R}^{m\times m}$ be a tri-diagonal matrix resulting from the application of m steps of Lanczos algorithm to A starting from $r_0 = b - Ax_0$, where $x_0 \in \mathbb{R}^n$ is given. Further let V_m be the matrix containing the orthonormal basis for Krylov subspace $\mathcal{K}_m(A, r_0)$ computed by the same algorithm.

a) Let $Q_m R_m = T_m$ be a QR-factorization of T_m. Show that R_m can be selected to be tri-diagonal and that it can be computed using $O(m)$ operations with the help of Givens rotations.

We now consider a projection method with the search and constraint spaces given by $\mathcal{K} = \mathcal{L} = \mathcal{K}_m(A, r_0)$.

b) Assuming that T_m is non-singular, show that $x_m = x_0 + V_m Q_m R_m^{-T}\|r_0\|_2 e_1$, where $e_1 \in \mathbb{R}^m$ is the first canonical basis vector and $Q_m R_m = T_m$ is the QR-factorization computed in **a)**.

Let $Q_{m-1}R_{m-1} = T_{m-1}$ be a QR-factorization of T_{m-1} computed in **a)**; thus Q_{m-1} is a product of some Givens rotations. Recall that T_{m-1} is a submatrix of T_m:

$$T_m = \begin{pmatrix} & & & 0 \\ & \Large T_{m-1} & & \vdots \\ & & & 0 \\ & & & \beta_m \\ 0 & \dots & 0 \quad \beta_m & \alpha_m \end{pmatrix}$$

c) Show that Q_m and R_m may be inexpensively computed from Q_{m-1} and R_{m-1}. Namely, show that for $m > 1$ we have the relations

$$Q_m = \begin{pmatrix} & & & 0 \\ & \Large Q_{m-1} & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} G_{m-1,m}^T,$$

where $G_{m-1,m}$ is some Givens rotation acting on dimensions $m-1$ and m, and that

$$R_m = \begin{pmatrix} \tilde{R}_{m-1} & r_m \\ 0 \quad \dots \quad 0 & \end{pmatrix},$$

where $\tilde{R}_{m-1}$ differs from R_{m-1} only at the bottom right element (i.e., the element in row $m-1$ and column $m-1$), and r_m is the last column of R_m. Show that only two Givens rotations are needed to compute $\tilde{R}_{m-1}$ and r_m given R_{m-1}, α_m, and β_m.

Problem 4 Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix and consider its splitting $A = A_1 - A_2$ with a non-singular A_1. For a given vector $b \in \mathbb{R}^n$ the corresponding matrix-splitting iterative method is defined by

$$A_1 x^{(k+1)} = A_2 x^{(k)} + b. \tag{2}$$

We will consider two versions of preconditioners based on matrix-splitting.

a) Let ν be a fixed positive integer, and the application of a preconditioner M^{-1} be equivalent to performing ν iterations of (2) starting from $x^{(0)} = 0$. Write down the algebraic expression for the right-preconditioned matrix AM^{-1}.

b) Let $\epsilon > 0$ be given, and assume that the iteration (2) converges for any starting point and any right hand side. We modify the definition of the preconditioner in **a)** as follows: the iteration (2) is repeatedly applied until the condition $\|A_1 x^{(k+1)} - A_2 x^{(k)} - b\|_2 \le \varepsilon \|b\|_2$ is satisfied.

Will the algebraic expression for the preconditioner remain constant from one iteration of, say, GMRES, to another?

Give an example of a Krylov subspace method that is specifically designed with such preconditioning strategies in mind.