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Solutions to exercise set 3

$\boxed{1}$ We are given the matrix

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 6 & 2 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

We may here use Gershgorin's theorem to estimate the eigenvalues of A. This theorem states that all the eigenvalues of an $n \times n$ matrix A are located in one of the closed discs of the complex plane centered in $a_{i,i}$ having radius

$$r_i = \sum_{\substack{j=1 \\ j\neq i}}^{j=n} |a_{i,j}|, \quad i = 1, \ldots, n.$$

See Saad, Theorem 4.6. For our matrix, this is illustrated in Figure 1. The shaded square encapsulates all the circles, and thus also the eigenvalues. For our given matrix, the spectrum is $\sigma(A) = \{2.328, 1.336 \pm 5.196\mathrm{i}\}$. We see that this is in agreement with the estimate.

Alternatively, one may use an estimate provided by Theorem 1.35 in Saad. That is, let us define the symmetric part of A

$$H = (A + A^{\mathrm{T}})/2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

and an anty-symmetric one

$$S = (A - A^{\mathrm{T}})/(2i) = \begin{bmatrix} 0 & 6i & 0 \\ -6i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

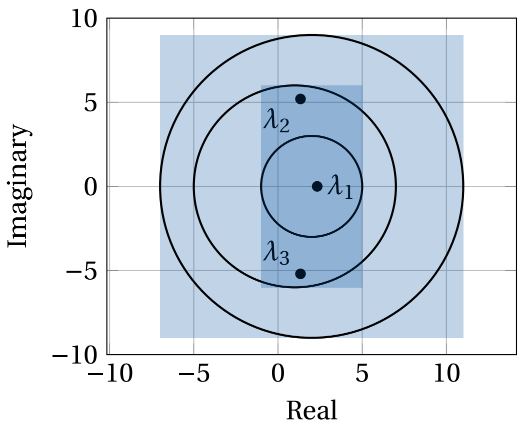

Figure 1: Exact and estimated eigenvalues of A.

Eigenvalues of H are $\{-1,1,5\}$ and eigenvalues of S are $\{-6,0,6\}$. Therefore, all eigenvalues of A lie in the rectangle $[-1,5]\times[-6,6]$, which provides a much sharper estimate (at additional computational cost) in this case; see Figure 1.

$\boxed{2}$ We follow the directions from Saad, Section 1.12.2. We form the matrices $V=(2,1)^{\mathrm{T}}$ containing a basis for Ran$P$ and $W=(1,-1)^{\mathrm{T}}$ containing a basis for the subspace $L=(\ker P)^{\perp}$. Then

$$P = V(W^{\mathrm{T}}V)^{-1}W^{\mathrm{T}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} [1]^{-1} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

P is not orthogonal because the computed matrix is not Hermitian. Alternatively, this can be immediately seen from the fact that Ran$P \not\perp \ker P$.