Norwegian University of Science and Technology
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TMA4205 Numerical Linear Algebra Fall 2015

Solutions to exercise set 8

1 **a)** Direct computation: $y^1 = A_1^{-1}v$, $y^2 = A_1^{-1}(A_2 y^1 + v) = A_1^{-1}(A_2 A_1^{-1} + I)v$, $y^\nu = [A_1^{-1}\sum_{j=0}^{\nu-1}(A_2 A_1)^{-\nu}]v$, therefore $M^{-1} = A_1^{-1}\sum_{j=0}^{\nu-1}(A_2 A_1)^{-\nu}$.

b) Without further information about A_2 (does it commute with A_1, for example?) we can only claim that M^{-1} will be SPD for $\nu = 1$, that is, when $M^{-1} = A_1^{-1}$.

c) In this case the matrix has a constant diagonal, and therefore Jacobi preconditioning is equivalent to scaling the equations by a constant. This has no effect on Krylov space methods, see Exercise 4 from set 6.

2 A direct computation shows that after one CG iteration we have $y_1 = (v^T v)/(v^T A v)v$. Let λ_1 and λ_2 be two distinct eigenvalues of A and let further v_1, v_2 be the corresponding eigenvectors. We will assume that $v_1 \perp v_2$ and $\|v_i\| = 1$. If we choose $v = \alpha_1 v_1 + \alpha_2 v_2$ then

$$y_1 = \frac{|\alpha_1|^2 + |\alpha_2|^2}{\lambda_1|\alpha_1|^2 + \lambda_2|\alpha_2|^2}[\alpha_1 v_1 + \alpha_2 v_2], \tag{1}$$

which is non-linear with respect to α_1, α_2. In particular, y_1 is a non-linear function of v in this case.

3 Let

$$B_R = \begin{pmatrix} x & y \\ z & 0 \end{pmatrix}.$$

Then

$$AB_R = \begin{pmatrix} x+2z & y \\ 3x & 3y \end{pmatrix}$$

and the corresponding least-squares problem can be written as

$$\min_{x,y,z} \frac{1}{2}\left\| \mathscr{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - b \right\|_2^2,$$

where

$$\mathscr{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

One could e.g. solve the normal equations to find $[x, y, z]^T = (\mathscr{A}^T\mathscr{A})^{-1}\mathscr{A}^T b = [0.0, 0.3, 0.5]^T$.

In this case

$$AB_R = \begin{pmatrix} 1 & 0.3 \\ 0 & 0.9 \end{pmatrix},$$

which is indeed not very far from the identity matrix.

A similar computation shows that

$$B_L = \begin{pmatrix} 0 & 1/3 \\ 0.4 & 0 \end{pmatrix}, \qquad \text{and} \qquad B_L A = \begin{pmatrix} 1 & 0 \\ 0.4 & 0.8 \end{pmatrix}.$$