



1 We shall study some properties of positive definite matrices.

a) Prove that the following two statements are equivalent for a normal matrix $A \in \mathbb{R}^{n \times n}$:

- The matrix A is positive definite.
- All the eigenvalues of A are positive.

b) Prove that the condition number based on the Euclidean norm of a normal matrix A , can be written as

$$\kappa(A) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|},$$

where λ_i are the eigenvalues of A .

c) Prove that a matrix A is positive definite if and only if A^{-1} is positive definite.

d) Let $A \in \mathbb{R}^{n \times n}$ and let $x \in \mathbb{R}^n$. The expression

$$R(x) = \frac{x^T A x}{x^T x}$$

is called the Rayleigh quotient of A . In case A is normal and positive definite, show that

$$\lambda_1 \leq R(x) \leq \lambda_n \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

where λ_1 and λ_n are the smallest and largest eigenvalue of A , respectively.

2 Consider the matrix

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}.$$

- Is A normal?
- Find the eigenvalues and eigenvectors of A .
- Are the eigenvectors linearly independent? Are they orthogonal?
- Can A be diagonalized?
- Show that there exists a constant $\alpha > 0$ so that $u^T A u \geq \alpha \|u\|_2^2$ for all $u \in \mathbb{R}^2$. What is the largest possible value for α ?
- Is A positive definite?
- Find a Schur factorization for A .

3 We consider again the Poisson problem

$$\begin{aligned} -\Delta u &= f, & \text{in } \Omega &= (0, 1) \times (0, 1), \\ u &= 0, & \text{on } \partial\Omega. \end{aligned}$$

We discretize the system on a uniform grid with grid size $h = 1/n$ in each direction.

- a) Solve the resulting system of linear equations in MATLAB with three different methods:
 - i) The diagonalization method discussed in Einar Rønquist's note.
 - ii) LU-factorization with sparse matrices.
 - iii) Full LU-factorization without sparse matrices.
- b) Do the timings scale as expected?
- c) Calculate the Euclidean condition number $\kappa(A)$ of the discrete Laplacian A for various different values of n . How does $\kappa(A)$ scale with n ?
- d) How does $\kappa(A)$ scale for the one-dimensional Laplacian? Compare these results with the results from question c).