EXAMINATION IN NUMERICAL SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENCE METHODS

MONDAY, MAY 24, 2004
TIME 09:00-12:00
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Exercisie 1. We consider the differential equation

$$\begin{aligned} u_t &= u_{xx}, \quad 0 \le x \le 1,\ t \ge 0, \\ u(x,0) &= f(x), \quad 0 \le x \le 1, \\ u(0,t) &= g_0(t), \quad u(1,t) = g_1(t),\ t \ge 0. \end{aligned} \tag{1}$$

We introduce a rectangular mesh with the points (x_m, t_n) with $x_m = mh,\ 0 \le m \le M,\ h = 1/M,\ t_n = nk,\ n = 0, 1, \ldots$. The quantity U_m^n approximates $u_m^n = u(x_m, t_n)$. We present a method of the type

$$U_m^{n+\frac{1}{2}} = U_m^n + \frac{r}{2}(\Delta_x U_m^n - \nabla_x U_m^{n+\frac{1}{2}}), \tag{2}$$

$$U_m^{n+1} = U_m^{n+\frac{1}{2}} + \frac{r}{2}(\Delta_x U_m^{n+1} - \nabla_x U_m^{n+\frac{1}{2}}), \tag{3}$$

where $r = \frac{k}{h^2}$. Here $U_m^{n+\frac{1}{2}}$ is used as an approximation to $u(x_m, t_n + \frac{1}{2}k)$. Δ_x and ∇_x are respectively the forward- and backward-differences in the $x-$direction.

a) Describe how one explicit step is calculated from the schemes (2),(3). Use probably a stencil to make your explanation clearer.

It is possible to write the method above in the form

$$U_m^{n+1} = U_m^n + \frac{1}{4}(2r + r^2)\delta_x^2 U_m^{n+1} + \frac{1}{4}(2r - r^2)\delta_x^2 U_m^n. \tag{4}$$

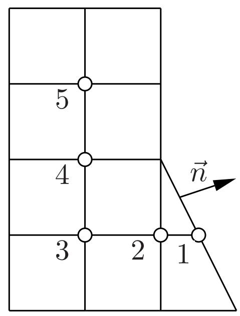
b) Investigate von Neumann criterion for the stability of this method.

c) It turns out that if we let $h \to 0$ and $k \to 0$ in such a way that $c = k/h$ stays constant, then the numerical approximation U_m^n will converge to $\tilde{u}(x_m, t_n)$ for all $x_m = mh$ and $t_n = nk$. But the function $\tilde{u}(x_m, t_n)$ is not an exact solution to the differential equation (1). Explain why this happens, and find a differential equation which has $\tilde{u}(x_m, t_n)$ as exact solution.

Exercise 2. We consider the Laplace equation on the domain Ω shown in the figure.

$$\Delta u = 0, \quad (x,y) \in \Omega,$$

with boundary data



$$\begin{aligned}
u(x,0) &= 0, & 0 \le x \le 0.75,\\
u(x,1) &= 0, & 0 \le x \le 0.5,\\
u(0,y) &= \sin \pi y, & 0 \le y \le 1,\\
u(0.5,y) &= \sin \pi y, & 0.5 \le y \le 1,\\
\frac{\partial u}{\partial n}\left(\frac{3-\lambda}{4}, \frac{\lambda}{2}\right) &= \lambda, & 0 \le \lambda \le 1,
\end{aligned}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$. As the figure suggests, stepsize $h = 0.25$ is used in both the $x-$ and $y-$directions.

Find a consistent discretization at the boundary node marked 1, and set up a linear system of equations for all the 5 unknowns in the figure.