EXAMINATION IN NUMERICAL SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENCE METHODS

MONDAY, JUNE 9, 2006
TIME 09:00-13:00
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Exercisie 1. In this exercise we condsider the advection-diffusion equation formulated as an initial-boundary value problem

$$\begin{aligned} u_t + au_x &= \nu u_{xx}, \qquad t > 0,\ 0 < x < 1, \\ u(x,0) &= f(x), \qquad 0 < x < 1, \\ u(0,t) &= g_0(t),\ u(1,t) = g_1(t),\ t \geq 0. \end{aligned} \tag{1}$$

a) We introduce a mesh with nodes (x_m, t_n) where $x_m = mh$, $0 \leq m \leq M$ and $t_n = nk$, $0 \leq n \leq N$ with stepsizes given as $h = 1/M$ and $k = T/N$. Let $r = k/h^2$ and $q = ak/h$. We denote by U_m^n an approximation to $u_m^n = u(x_m, t_n)$, and propose the following scheme for (1)

$$U_m^{n+1} = (\nu r + \frac{\alpha+1}{2}q)U_{m-1}^n + (1 - 2\nu r - \alpha q)U_m^n + (\nu r + \frac{\alpha-1}{2}q)U_{m+1}^n, \tag{2}$$

where α is a parameter we can choose. Find an expression for leading term in the truncation error in terms of α, and comment on the value of α that gives a better order of convergence.

b) Assume that we choose $\alpha = 0$ in (2), but replace the diffusion coefficient ν in (1) by $\bar{\nu} = \nu + \frac{1}{2}\alpha a h$. Write down the resulting difference formula, and comment on the result.

c) Put $\alpha = 0$ in (2) and investigate the stability of the method on the time interval $[0, T]$ via the matrix method.
Hint. The matrix A to be investigated is diagonalizable such that $A = T\Lambda T^{-1}$ with

$$\|T\|_2 \cdot \|T^{-1}\|_2 \leq e^{\frac{|a|}{2\nu}}, \qquad \text{for all } h, k.$$

Exercise 2. A function $u(x,y)$ satisfies the elliptic differential equation

$$u_{xx} + 3u_{yy} = -16$$

on the square bounded by the lines $x = \pm 1$ and $y = \pm 1$. The boundary conditions are $u = 0$ for $x = 1$ and $\frac{\partial u}{\partial y} = -1$ for $y = 1$. Moreover $u(x,y)$ is symmetric about both the $x-$axis and the $y-$axis. We use a $5-$point difference formula, and a uniform mesh with stepsizes $\Delta x = \Delta y = \frac{1}{4}$.

a) Show that the solution to this problem can be approximated by solving a system of equations of the form $AU = b$ where A is a $20 \times 20-$matrix of the form

$$A = \begin{bmatrix} B & 6I & & & \\ 3I & B & 3I & & \\ & 3I & B & 3I & \\ & & 3I & B & 3I \\ & & & 6I & B \end{bmatrix}$$

b) Find the matrix B.

Exercise 3. We consider the hyperbolic equation

$$u_t + 2tu_x = 0. \tag{3}$$

a) Show that the characteristic through the point $(X, T),\ T > 0$ is given by the equation

$$t = \sqrt{x - X + T^2}, \quad t > 0.$$

b) Hence determine the CFL-condition for an explicit scheme of the type

$$U_m^{n+1} = a_{-1}U_{m-1}^n + a_0 U_m^n + a_1 U_{m+1}^n$$

applied to (3).

c) Investigate the dissipative properties of the scheme

$$U_m^{n+1} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) - \frac{ak}{2h}(U_{m+1}^n - U_{m-1}^n)$$

for the equation $u_t + au_x = 0$.