Norwegian University of
Science and Technology
Department of Mathematical Sciences

Contact during the exam:
Bawfeh K. Kometa, tlf. 735 91972

EXAM IN TMA4212
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Time: 09:00–13:00

Allowed material: code B – All printed and handwritten material is allowed.
A simple calculator is allowed.

Problem 1

a) Consider the boundary value problem

$$-\varepsilon u_{xx} + au = 0 \tag{1}$$
$$u(0) = 0, \tag{2}$$
$$u(1) = 1, \tag{3}$$

$\varepsilon > 0$, $a > 0$ and $0 < x < 1$. Consider a mesh $x_0, \dots, x_M$ with $x_m = mh$ and $h = 1/M$, and denote with u_m the numerical approximation of $u(x_m)$. Discretize this problem with central differences, the linear system of algebraic equations you obtain is

$$A^h U = \mathbf{b}, \tag{4}$$

with

$$A^h = a\,I - \frac{\varepsilon}{h^2}\,\mathrm{tridiag}(1, -2, +1), \quad U = (u_1, \dots, u_{M-1})^T, \quad \mathbf{b} \in \mathbf{R}^{M-1}.$$

Find $\mathbf{b}$.[1]

Consider the local truncation error $\tau := A^h\mathbf{u} - \mathbf{b}$ where

$$\mathbf{u} := (u(x_1), \dots, u(x_{M-1}))^T,$$

is the exact solution of (1) tabulated at the nodes of the discretization mesh. The solution u of the boundary value problem is smooth bounded

[1] Here I is the identity matrix and $\mathrm{tridiag}(\alpha, \beta, \gamma)$ is the tridiagonal matrix with β on the main diagonal, α on the sub-diagonal and γ on the super-diagonal.

and with bounded derivatives on $[0,1]$. Using Taylor expansion show that the j-th component of the local truncation error is

$$\tau_j = -\frac{\varepsilon}{12} h^2 u_{xxxx}(x_j) - \mathcal{O}(h^4).$$

b) Consider now the error $E^h := U - \mathbf{u}$, and show that the method is convergent.

c) Show that

$$u_j = ({\tau_2}^M - {\tau_1}^M)^{-1}({\tau_2}^j - {\tau_1}^j), \quad j = 1, \dots, M-1,$$

with

$$\tau_1 = 1 + \sigma - \sqrt{\sigma(2+\sigma)}, \quad \tau_2 = 1 + \sigma + \sqrt{\sigma(2+\sigma)},$$

where $\sigma = ah^2/(2\varepsilon)$.

d) Consider a finite element discretization of (1) using piecewise-linear finite elements on the mesh $x_0, \dots, x_M = 1$ with $x_m = mh$ and $h = 1/M$. Use the bilinear form

$$a(u,v) = \varepsilon \int_0^1 u_x v_x \, dx + a \int_0^1 uv \, dx,$$

to write the Galerkin formulation of the problem using appropriate function spaces[2], assume $u \in H^1_E$ and $v \in H^1_0$. Explain the connection between the Galerkin formulation and (1).

Write now the Galerkin method and the corresponding linear system of equations

$$(C + B)U = b,$$

where C is the stiffness matrix and B is the mass matrix. Recall that the mass matrix B is the $(M-1) \times (M-1)$ matrix with entries

$$B_{i,j} := \int_0^1 \varphi_i \varphi_j \, dx, \quad i,j = 1, \dots, M-1,$$

and $\varphi_1, \dots, \varphi_{M-1}$ are the finite element basis functions. Use exact integration to find the entries of C and B.

e) By choosing appropriate quadrature modify the mass matrix B into a new matrix $\hat{B}$ so that the resulting method gives the same numerical solution obtained from (4).

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[2]

$$\begin{aligned}
H^1((0,1)) &:= \{v \in L_2((0,1)) \,|\, v \,\text{absolutely continuous on}\, [0,1],\ \partial x v \in L_2((0,1))\,\}, \\
H^1_0((0,1)) &:= \{v \in H^1((0,1)) \,|\, v(0) = v(1) = 0\,\}, \\
H^1_E((0,1)) &:= \{v \in H^1((0,1)) \,|\, v(0) = 0,\ v(1) = 1\,\}.
\end{aligned}$$

Problem 2

a) Consider the linearized Korteweg-deVries equation (KdV), as a pure initial value problem

$$u_t + \rho u_x + u_{xxx} = 0, \quad u(-\infty) = u(\infty) = 0, \quad x \in \mathbf{R},$$

ρ a given real constant.

Consider the interval $[-L, L]$ with $L > 0$ sufficiently large, consider the grid $x_m = -L + hm$, $h = 2L/M$, $m = 0, \dots, M$. Discretize the problem with central finite differences in space and with the trapezoidal rule in time (the Crank-Nicolson method), let $u(x_0, t) = u(x_M, t) = 0$.

Use the following central differences approximation of the third derivative

$$u_{xxx}|_{x_m} = \frac{u(x_{m+3}) - 3u(x_{m+1}) + 3u(x_{m-1}) - u(x_{m-3})}{8h^3} + \mathcal{O}(h^2).$$

Show that the obtained method is Von Neumann stable.

b) Consider the energy function

$$H(u(t)) = \frac{1}{2}\int_{-\infty}^{\infty} {u_x}^2\, dx - \frac{\rho}{2}\int_{-\infty}^{\infty} u^2\, dx.$$

Assume u and all its derivatives with respect to x vanish for $|x| \to \infty$. Using integration by parts, show that H is constant along solutions of the linearized KdV equation (i.e. $\frac{dH(u(t))}{dt} = 0$).

c) Assume now that D is the skew-symmetric matrix corresponding to the central finite difference discretization of the first derivative[3]. It is easy to verify that the matrix A corresponding to the central difference discretization of the third derivative is such that $A = D^3$. Write the Crank-Nicolson method in the form $U^{n+1} = CU^n$ for an appropriate matrix C and with $U^n = (u_1^n, \dots, u_{M-1}^n)^T$ the numerical solution. Show that the discrete energy function

$$H \approx \tilde{H}(U) := \frac{1}{2}\|DU\|_2^2 - \frac{\rho}{2}\|U\|_2^2, \quad U \in \mathbf{R}^{M-1}$$

is constant along numerical solutions given by the Crank-Nicolson method, that is $\tilde{H}(U^{n+1}) = \tilde{H}(U^n)$. Here $\|\cdot\|_2$ is the grid-function-norm and is an approximation of the integrals defining H.

[3] Skew-symmetric matrices are diagonalizable with pure imaginary eigenvalues and a set of orthonormal eigenvectors: $D = V\Lambda V^H$ with λ a diagonal matrix with diagonal elements $\lambda_m = i\alpha_m$, $m = 1, \dots, M-1$, with $i = \sqrt{-1}$, α_m real, and $V^H V = I$. V^H is the conjugate-transpose of V.

Problem 3

a) Consider the partial differential equation

$$u_t = u_{xx} - \gamma u, \quad 0 < x < 1, \quad t \in [0, T], \quad \gamma > 0,$$

with Neumann boundary conditions

$$-\partial_x u|_{x=0} = 0, \ \partial_x u|_{x=1} = 1.$$

We discretize in space using the method of line discretizations. We use a second order discretization of the boundary conditions, and a central difference discretization of the second derivative with respect to x, and we obtain a linear system of ordinary differential equations of the type

$$\dot{U} = AU + g, \tag{5}$$

where $U(t) := (u_0(t), \ldots, u_M(t))^T$ and $u_m(t) \approx u(x_m, t)$, $m = 0, \ldots, M$.

Find A and g.

b) We take the exact solution of the linear system of ordinary differential equations (5) as numerical approximation of the solution of partial differential equation, that is

$$(u_0^n, \ldots, u_M^n)^T := (u_0(t_n), \ldots, u_M(t_n))^T, \quad U^n := U(t_n), \tag{6}$$

and $u_m^n \approx u(x_m, t^n)$.

Consider the exact solution tabulated at the nodes of the discretization and at time t, $\mathbf{u}(t) = (u(x_0, t), \ldots, u(x_M, t))^T$, the local truncation error is

$$\tau := \dot{\mathbf{u}} - A\mathbf{u} - g,$$

and you may use that $\|\tau\|_2 = Ch^2 + \mathcal{O}(h^4)$.

Prove convergence of the finite difference scheme.

Extra information

Assume g is a constant vector, A is invertible and I is the identity matrix[4]. One can solve (5) exactly by the variation of constants formula to obtain

$$U(t) = \exp((t - t_0)A)U(t_0) + f(t - t_0, A)\, g, \tag{7}$$

with

$$f(t - t_0, A) = A^{-1}(\exp((t - t_0)A) - I), \quad \exp(sA) := \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k.$$

[4]The assumptions g constant and A invertible are not essential to apply the variation of constants formula, but are useful to simplify the formulae.

Here $\exp(sA)$ is the matrix exponential and if A is diagonalizable, also $\exp(sA)$ is diagonalizable,

$$A = X\Lambda X^{-1} \Rightarrow \exp(sA) = X\exp(s\Lambda)X^{-1}.$$

You can use (7) to write the numerical approximation U^n of (6) and to obtain $U^{n+1} = CU^n + q$, with appropriate C and q.

Piecewise-linear finite element functions

$$\phi_j(x) = \begin{cases} \frac{(x-x_{j-1})}{h} & x_{j-1} \le x \le x_j, \\ \frac{(x_{j+1}-x)}{h} & x_j \le x \le x_{j+1}, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1, \dots, M-1,$$

$$\phi_M(x) = \begin{cases} \frac{(x-x_{M-1})}{h} & x_{M-1} \le x \le x_M, \\ 0 & \text{otherwise}, \end{cases} \qquad \phi_0(x) = \begin{cases} \frac{(x_0-x)}{h} & x_0 \le x \le x_1, \\ 0 & \text{otherwise}. \end{cases}$$

Eigenvalues of the discrete Laplacian

Consider the $(M-1)\times(M-1)$ matrix $B^h = 1/h^2\,\text{tridiag}(1,-2,1)$, discretization of the Laplacian (with homogeneous Dirichlet boundary conditions). The eigenvalues of B^h are

$$\lambda_p(B^h) = 2/h^2\,(\cos(p\pi h) - 1), \quad p = 1, \dots, M-1.$$