TMA4212 Numerical solution of partial differential equations with finite difference methods

Problem Set 1

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix with constant diagonals,

$$A = \begin{pmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{pmatrix}$$

where $bc > 0$. Let $D = \operatorname{diag}(1, \alpha, \alpha^2, \dots, \alpha^{n-1})$. Show that there exists (find) an α so that $S = D^{-1}AD$ is symmetric. What does S look like?

Problem 2. The p-norm of a vector $x \in \mathbb{R}^n$ is given by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Show that if $D = \operatorname{diag}(d_1, \dots, d_n)$, then the associated matrix norm of D is given by

$$\|D\|_p = \max_{1\le i\le n} |d_i| = \rho(D),$$

where $\rho(D)$ is the spectral radius of D.

Problem 3.

a) Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$, and let T be an invertible $n \times n$ matrix. Show that the function

$$f_T(x) = \|T^{-1}x\|$$

defines a vector norm on $\mathbb{R}^n$. We then write $\|x\|_T = f_T(x) = \|T^{-1}x\|$ for this norm.

b) We define the matrix norm associated with the vector norm $\|\cdot\|$ as usual with

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

Show that the matrix norm associated with the vector norm $\|\cdot\|_T$ is

$$\|A\|_T = \|T^{-1}AT\|.$$

c) In the classes we learned that the spectral radius $\rho(A)$ of a matrix A satisfies

$$\rho(A) \le \|A\|$$

for every matrix norm $\|\cdot\|$. Suppose A is fixed. Show that for every $\epsilon > 0$ there is a matrix norm $\|\cdot\|_{A,\epsilon}$ such that

$$\|A\|_{A,\epsilon} \le \rho(A) + \epsilon.$$

Hint. Use the Jordan form of A to modify the p-norm as in **b)**. Modify this norm further by using a diagonal matrix in the same form as D from problem 1.