TMA4212 Numerical solution of partial differential equations with finite difference methods

Solutions to Problem Set 1

Problem 1. *Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix with constant diagonals,*

$$A = \begin{pmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{pmatrix}$$

where $bc > 0$. Let $D = \operatorname{diag}(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$. Show that there exists (find) an α so that $S = D^{-1}AD$ is symmetric. What does S look like?

Solution. Note that $D^{-1} = \operatorname{diag}(1, \alpha^{-1}, \alpha^{-2}, \ldots, \alpha^{-n+1})$. We then see that

$$S_{ij} = (D^{-1}AD)_{ij} = \alpha^{j-i}A_{ij},$$

so S is also tridiagonal with constant diagonals. We must require that $S_{i,i+1} = S_{i+1,i}$, which means

$$\alpha b = \alpha^{-1}c \qquad \Longrightarrow \qquad \alpha = \sqrt{c/b},$$

which is fine since $bc > 0$. Then $S_{i,i+1} = S_{i+1,i} = \sqrt{bc}$, so S is the tridiagonal and symmetric matrix $S = \operatorname{trid}(\sqrt{bc}, a, \sqrt{bc})$.

Problem 2. *The p-norm of a vector $x \in \mathbb{R}^n$ is given by*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Show that if $D = \operatorname{diag}(d_1, \ldots, d_n)$, then the associated matrix norm of D is given by

$$\|D\|_p = \max_{1\le i\le n} |d_i| = \rho(D),$$

where $\rho(D)$ is the spectral radius of D.

Solution. For arbitrary x,

$$\|Dx\|_p = \left(\sum_i |d_i x_i|^p\right)^{1/p} \le \max_i |d_i| \left(\sum_i |x_i|^p\right)^{1/p} = \rho(D)\|x\|_p,$$

since the eigenvalues of a diagonal matrix are on the diagonal. Then we see that $\|D\|_p \le \rho(D)$. If $D = 0$, the statement is trivially true, so suppose $D \le 0$, and let k be so that $\max_i |d_i| = |d_k| = \rho(D) > 0$. Now let

$$x = \frac{|d_k|}{d_k} e_k \qquad \Longrightarrow \qquad \|Dx\|_p = |d_k| = \rho(D) = \rho(D)\|x\|_p$$

since $\|x\|_p = 1$, and we conclude that $\|D\|_p = \rho(D)$.

Problem 3.

a) *Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$, and let T be an invertible $n \times n$ matrix. Show that the function*

$$f_T(x) = \|T^{-1}x\|$$

defines a vector norm on $\mathbb{R}^n$. We then write $\|x\|_T = f_T(x) = \|T^{-1}x\|$ for this norm.

Solution. We must check the axoims for norms, assuming that $\|\cdot\|$ already satisfies them.

1. Clearly $f_T(x) = \|T^{-1}x\| > 0$ and $f_T(x) = 0 \Rightarrow T^{-1}x = 0 \Rightarrow x = 0$.
2. $f_T(\alpha x) = \|T^{-1}\alpha x\| = |\alpha|\|T^{-1}x\| = |\alpha| f_T(x)$.
3. $f_T(x+y) = \|T^{-1}(x+y)\| = \|T^{-1}x + T^{-1}y\| \leq \|T^{-1}x\| + \|T^{-1}y\| = f_T(x) + f_T(y)$.

b) *We define the matrix norm associated with the vector norm $\|\cdot\|$ as usual with*

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Show that the matrix norm associated with the vector norm $\|\cdot\|_T$ is

$$\|A\|_T = \|T^{-1}AT\|.$$

Solution. By definition,

$$\|A\|_T = \sup_{x \neq 0} \frac{\|Ax\|_T}{\|x\|_T} = \sup_{x \neq 0} \frac{\|T^{-1}Ax\|}{\|T^{-1}x\|}.$$

Now let $y = T^{-1}x$, and note that as x runs through $R^n \backslash \{0\}$, so will y. For this reason, we can take the supremum over y instead, to get

$$\|A\|_T = \sup_{y \neq 0} \frac{\|T^{-1}ATy\|}{\|y\|} = \|T^{-1}AT\|.$$

c) *In the classes we learned that the spectral radius $\rho(A)$ of a matrix A satisfies*

$$\rho(A) \leq \|A\|$$

for every matrix norm $\|\cdot\|$. Suppose A is fixed. Show that for every $\epsilon > 0$ there is a matrix norm $\|\cdot\|_{A,\epsilon}$ such that

$$\|A\|_{A,\epsilon} \leq \rho(A) + \epsilon.$$

Hint. *Use the Jordan form of A to modify the p-norm as in* ***b)****. Modify this norm further by using a diagonal matrix in the same form as D from problem 1.*

Solution. Let $A = TJT^{-1}$ be a Jordan form factorization of A. From the above we know that $\|A\|_T = \|T^{-1}AT\| = \|J\|$. The Jordan matrix J has the eigenvalues of A on the diagonal, and possibly some 1's on the first superdiagonal. Now with

$$D = \operatorname{diag}(1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}),$$

we construct the norm

$$\|A\|_{TD} = \|D^{-1}JD\|.$$

As in problem 1, we see that $(D^{-1}JD)_{ii} = J_{ii}$ and $(D^{-1}JD)_{i,i+1} = \epsilon J_{i,i+1}$, where $J_{i,i+1}$ is either 0 or 1. All other elements are zero. Let us choose the base norm $\|\cdot\| = \|\cdot\|_1$, which we remember is maximal column sum. Then we get

$$\|A\|_{TD} = \|D^{-1}JD\|_1 \leq \rho(A) + \epsilon,$$

so this is the norm we seek.