TMA4212 Numerical solution of differential equations with difference methods

Problem Set 2

Problem 1. Write a MATLAB program that solves the following problem with the Crank–Nicolson method.

$$u_t = u_{xx}, \quad 0 < x < 1, \; t > 0$$

$$u(x,0) = 0, \quad 0 \le x \le 1$$

$$u(0,t) = \begin{cases} 10t, & 0 \le t \le 0.1 \\ 1, & t > 0.1 \end{cases}$$

$$u(1,t) = 0$$

Use steplengths $h = 0.1$ and $k = 0.01$ up to $t = 0.1$, and $k = 0.1$ from there.

Assume that the exact solution $u(x,t)$ approaches a stationary value $u_\infty(x)$ when t grows large. Find $u_\infty(x)$. Investigate, by studying the output from your program, whether the numerical solution U_m^n approaches $u_\infty(x_m)$ as $n \to \infty$. Can you estimate the error?

Problem 2. When we model a physical problem, we often start with element considerations, and upon letting the element sizes approach zero, we arrive at a partial differential equation. In a way this is a numerical algorithm in reverse. Let us see an example.

We consider heat balance in a thin rod with length L and varying cross-section $F(x)$. We split the rod into M pieces along the x-axis, each with length $h = L/M$. We form volume elements, each one bounded by the cross sections at $x_{m\pm1/2} = (m \pm 1/2)h$. We assume that the temperature $T_m(t)$ is constant over the element. To form the heat balance for each element, we need to consider transport of heat to and from the neighboring elements, and transport of heat to the surrounding air. We assume that the rod has density ρ, specific heat capacity c, conductivity λ and convection coefficient α towards air, all independent of x. The rods circumference $R(x)$, the end temperatures T_v and T_r, the intial temperature as well as the air temperature T_A are assumed known. We put $F_{m\pm1/2} = F(x_{m\pm1/2})$ and $R_m = R(x_m)$. The heat balance for an element is then approximately

$$\begin{aligned} \rho c \tfrac{h}{2}(F_{m+1/2} + F_{m-1/2})\tfrac{dT_m}{dt} = \; & -\underbrace{\lambda F_{m-1/2}(T_m - T_{m-1})/h}_{\text{flux at } x_{m-1/2}} + \underbrace{\lambda F_{m+1/2}(T_{m+1} - T_m)/h}_{\text{flux at } x_{m+1/2}} \\ & -\underbrace{\alpha h R_m (T_m - T_A)}_{\text{heat to air}} \end{aligned}$$

We have used $\frac{h}{2}(F_{m+1/2} + F_{m-1/2})$ as an approximation to the volume of the element, and hR_m as an approximation to its surface area. Letting $m = 1, \ldots, M-1$, we get a system of ODEs for $T_m(t)$. These equations may be considered as a semidiscretization of a PDE for a function $T(x,t)$. Your task is to find this PDE.

Problem 3. Does the following argument hold?

$$\begin{aligned} \partial_t u = \partial_x^2 u \; &= \tfrac{1}{h^2}\delta_x^2 u - \tfrac{1}{12}h^2 \partial_x^4 u + \mathcal{O}(h^4) \\ &= \tfrac{1}{h^2}\delta_x^2 u - \tfrac{1}{12}h^2 \partial_x^2 \partial_t u + \mathcal{O}(h^4) \\ &= \tfrac{1}{h^2}\delta_x^2 u - \tfrac{1}{12}\delta_x^2 \partial_t u + \mathcal{O}(h^4) \end{aligned}$$

thus

$$(1 + \frac{1}{12}\delta_x^2)\partial_t u = \frac{1}{h^2}\delta_x^2 u + \mathcal{O}(h^4)$$

so by the trapezoidal rule

$$(1+\frac{1}{12}\delta_x^2)(u_m^{n+1}-u_m^n)=\frac{k}{2h^2}(\delta_x^2u_m^n+\delta_x^2u_m^{n+1})+\mathcal{O}(kh^4)+\mathcal{O}(k^3).$$

In which case

$$\left(1+(\frac{1}{12}-\frac{r}{2})\delta_x^2\right)U_m^{n+1}=\left(1+(\frac{1}{12}+\frac{r}{2})\delta_x^2\right)U_m^n$$

(where $r=k/h^2$) is a difference scheme with local truncation error $\mathcal{O}(k^3+kh^4)$. Compute the principal term of the truncation error.

Problem 4. Compute the truncation error in

$$\frac{1}{h^2}\delta\,(a(x)\cdot\delta\,u(x))$$

as an approximation to $(a(x)u'(x))'$.