TMA4212 Numerical solution of partial differential equations with finite difference methods

Solutions to Problem Set 2

Problem 1. The Crank–Nicolson method for the equations $u_t = u_{xx}$ can be written

$$\left(1 - \frac{r}{2}\delta_x^2\right) U_m^{n+1} = \left(1 + \frac{r}{2}\delta_x^2\right) U_m^n,$$

where U_m^n is an approximation to $u(nk, mh)$ and $r = k/h^2$. We will use two different steplengths along the time axis, $k_1 = 0.01$ and $k_2 = 0.1$, and a fixed steplength $h = 1/M = 0.1$ along the x-axis. We construct a solution vector

$$U^n = \left[U_1^n, U_2^n, \ldots, U_{M-1}^n\right]^T,$$

after which the method can be written in matrix form,

$$AU^{n+1} = BU^n + g^n = c^n,$$

where

$$A = \begin{bmatrix} 1+r & -r/2 & 0 & \cdots & \cdots & 0 \\ -r/2 & 1+r & -r/2 & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & -r/2 & 1+r & -r/2 \\ 0 & \cdots & \cdots & 0 & -r/2 & 1+r \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1-r & r/2 & 0 & \cdots & \cdots & 0 \\ r/2 & 1-r & r/2 & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & r/2 & 1-r & r/2 \\ 0 & \cdots & \cdots & 0 & r/2 & 1-r \end{bmatrix},$$

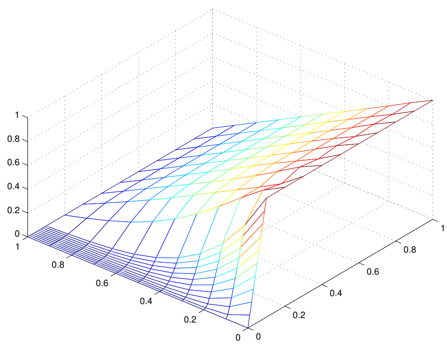
and g^n fixes the boundary conditions,

$$g^n = \left[r(U_0^{n+1} + U_0^n)/2, 0, \ldots, 0\right]^T,$$

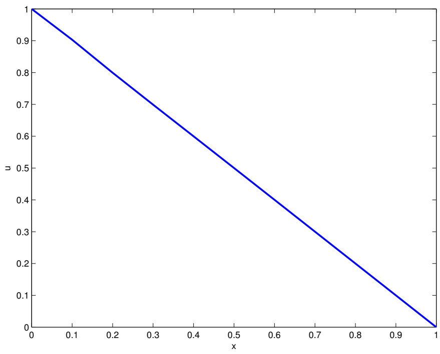
where $U_0^n = u(0, nk)$, and we also define $U_M^n = 0$.

When implementing this in MATLAB, we should appreciate that the matrices involved are tridiagonal and thus highly sparse. We can generate these matrices using a function such as `spdiags`, and we can use Gauss-elimination (or an even more efficient tridiagonal algorithm) to solve the system $AU^{n+1} = c^n$. A program to do this is given below. The total solution is stored in a matrix `Utot`, where `Utot[m,n]`$= U_{m-1}^{n-1}$. Figure 1 shows the numerical solution as a function of x and t. The solution appears to approach $1 - x$ as t grows. We can compare this with the exact stationary solution $u_\infty(x)$. We must solve

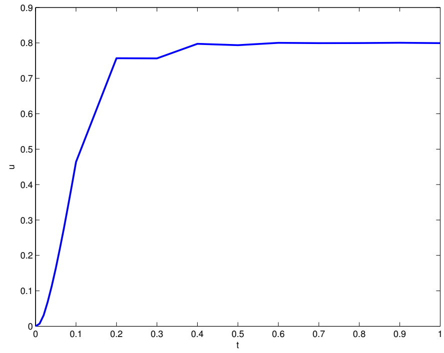
$$\frac{\partial u_\infty(x)}{\partial x^2} = 0, \quad u_\infty(0) = 1, \quad u_\infty(1) = 0.$$



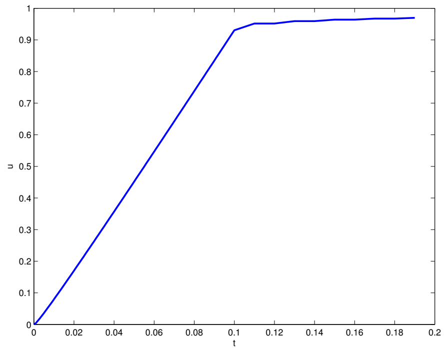
(a) Numerical solution wrt. x and t.



(b) A cross-section for $t = 1$.



(c) A cross-section for $x = 0.2$.



(d) A cross-section for $x = 0.2$, but with $M = 100$, $k_1 = 0.001$ and $k_2 = 0.01$.

Figure 1: Plots.

```
M=10;  k1=0.01;  k2=0.1;                 % Resolution
h=1/M;  r1=k1/h^2;  r2=k2/h^2;
N1=0.1/k1;  N=N1+9;                      % Steps in time

tid=[0:k1:0.1  0.1+k2:k2:0.1+(N-N1)*k2]; % Time-axis
x=0:h:1;                                 % X-axis

U0=ones(1,N);                            % Boundary values
for n=1:N1
   U0(n)=10*n*k1;
end
UM=zeros(1,N);
Ustart=zeros(M-1,1);                     % Initial condition

% Matrix generation

d=ones(M-1,1);
A=spdiags([-r1*d/2 (1+r1)*d -r1*d/2],-1:1,M-1,M-1);
B=spdiags([r1*d/2 (1-r1)*d r1*d/2],-1:1,M-1,M-1);

U=zeros(M-1,N);                  % Allocating memory for solution
c=B*Ustart;                      % First c-vector
c(1)=c(1)+r1/2*U0(1);            % Boundary contribution
U(:,1)=A\c;                      % First time step
for n=1:N1-1                     % First set of iterations
   c=B*U(:,n);
   c(1)=c(1)+r1/2*(U0(n)+U0(n+1));
   U(:,n+1)=A\c;
end

% New matrices (new k)

A=spdiags([-r2*d/2 (1+r2)*d -r2*d/2],-1:1,M-1,M-1);
B=spdiags([r2*d/2 (1-r2)*d r2*d/2],-1:1,M-1,M-1);

for n=N1:N-1                     % Second set of iterations
   c=B*U(:,n);
   c(1)=c(1)+r2/2*(U0(n)+U0(n+1));
   U(:,n+1)=A\c;
end

Utot=[[0 U0];[Ustart U];[0 UM]];    % Total solution
mesh(tid,x,Utot);                % Plotting
view(-50,50);
```

which yields precisely $u_\infty(x) = 1 - x$.

A closer look at figure 1 reveals some small oscillations, made more obvious by a plot along $x = 0.2$. These are typical for Crank–Nicolson, but can be improved by increasing the resolution.

Problem 2. We consider the limit of the semidiscrete equation as $h \to 0$ (and $m \to \infty$ so that $mh = x$ is constant). After dividing the equation by h, we find the following "inverse discretization" for the relevant terms (F and T are assumed continuous and differentiable):

$$\begin{aligned}
(F_{m+1/2} + F_{m-1/2})/2 \quad &\to \quad F(mh) = F(x) \\
(T_m - T_{m-1})/h \quad &\to \quad \left(\frac{\partial T}{\partial x}\right)_{m-1/2} \\
\frac{1}{h}\left[F_{m+1/2}\left(\frac{\partial T}{\partial x}\right)_{m+1/2} - F_{m-1/2}\left(\frac{\partial T}{\partial x}\right)_{m-1/2}\right] \quad &\to \quad \frac{\partial}{\partial x}\left(F(x)\frac{\partial T(x,t)}{\partial x}\right).
\end{aligned}$$

The PDE then becomes

$$\begin{aligned}
\rho c F(x)\frac{\partial T(x,t)}{\partial t} &= \lambda\frac{\partial}{\partial x}\left[F(x)\frac{\partial T(x,t)}{\partial x}\right] - \alpha R(x)[T(x,t) - T_A], \\
&T(0,t) = T_v, \quad T(L,t) = T_h, \quad T(x,0) \text{ given.}
\end{aligned}$$

Problem 3. We will try to justify the deduction given in the problem text. By Taylor expansion we have

$$\delta_x^2 u = h^2\partial_x^2 u + \frac{1}{12}h^4\partial_x^4 u + O(h^6) \quad \Longleftrightarrow \quad \partial_x^2 u = \frac{1}{h^2}\delta_x^2 u - \frac{1}{12}h^2\partial_x^4 u + O(h^4). \tag{1}$$

This can be inserted into the equation $\partial_t u = \partial_x^2 u$, giving the first line of the problem. Since $\partial_x^4 u = \partial_x^2\partial_t u$, we can use (1) again. The term $\frac{1}{144}h^4\partial_x^4\partial_t u$ is included in the $O(h^4)$-term, yielding line 4.

By operating with $(1 + \frac{1}{12}\delta_x^2)^{-1}$ on both sides of line 4, we arrive at

$$u_t = (1 + \frac{1}{12}\partial_x^2)^{-1}\frac{1}{h^2}\delta_x^2 u + O(h^4) \equiv Du + O(h^4),$$

which by the trapezoidal rule gives

$$\frac{1}{k}(u_m^{n+1} - u_m^n) = \frac{1}{2}D(u_m^n + u_m^{n+1}) + O(h^4).$$

By multiplying with $1 + \frac{1}{12}\delta_x^2$, we reach the expression and the method given in the problem. The truncation error is then given by

$$T_m^n = \left[1 + \left(\frac{1}{12} - \frac{r}{2}\right)\delta_x^2\right]u_m^{n+1} - \left[1 + \left(\frac{1}{12} + \frac{r}{2}\right)\delta_x^2\right]u_m^n. \tag{2}$$

Using the PDE and Taylor expansion we have (denoting $u \equiv u_m^n$, $u_t \equiv (u_t)_m^n$ etc.):

$$\begin{aligned}
u_m^{n+1} &= u + ku_t + \frac{1}{2}k^2u_{tt} + \frac{1}{6}k^3u_{3t} + O(k^4), \\
\delta_x^2 u_m^n &= h^2u_t + \frac{1}{12}h^4u_{tt} + \frac{1}{360}h^6u_{3t} + O(h^8), \\
\delta_x^2 u_m^{n+1} &= h^2u_t + \left(\frac{1}{12}h^4 + kh^2\right)u_{tt} + \left(\frac{1}{360}h^6 + \frac{1}{12}kh^4 + \frac{1}{2}k^2h^2\right) \\
&\quad + O(k^4 + k^3h^2 + k^2h^4 + kh^6 + h^8).
\end{aligned}$$

Inserting this into (2) and cancelling a *lot*, we arrive at

$$T_m^n = \left(-\frac{1}{12}k^3 + \frac{1}{240}kh^4\right)u_{3t} + O(k^4 + k^3h^2 + k^2h^4 + kh^6 + h^8).$$

Problem 4. By Taylor expansion,

$$\begin{aligned}
T_n &= \frac{1}{h^2}\delta(a_n\delta u_n) - (a_n u_n')' = \frac{1}{h^2}\left[a_{n+1/2}(u_{n+1} - u_n) - a_{n-1/2}(u_n - u_{n-1})\right] - a_n' u_n' - a_n u_n'' \\
&= \frac{1}{h^2}\left(a + \frac{h}{2}a' + \frac{h^2}{8}a'' + \frac{h^3}{48}a'''\right)\left(hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \frac{h^4}{24}u''''\right) - \\
&\quad \left(a - \frac{h}{2}a' + \frac{h^2}{8}a'' - \frac{h^3}{48}a'''\right)\left(hu' - \frac{h^2}{2}u'' + \frac{h^3}{6}u''' - \frac{h^4}{24}u''''\right) + O(h^4) - a'u' - au'' \\
&= \left(\frac{1}{12}au'''' + \frac{1}{6}a'u''' + \frac{1}{8}a''u'' + \frac{1}{24}a'''u'\right)h^2 + O(h^4).
\end{aligned}$$