TMA4212 Numerical solution of partial differential equations with finite difference methods

Solutions to Problem Set 4

Problem 1. a)

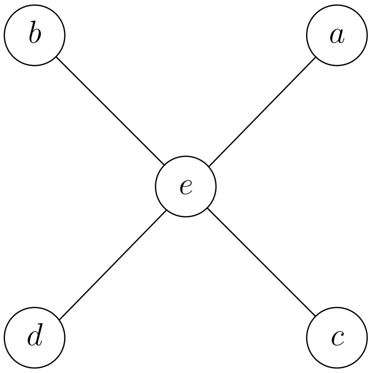

Figure 1: The calculation molecule.

See figure 1.

b) The first thing we can see is that e will be on the diagonal. We want two subdiagonals where d and c are placed $M+1$ and $M-1$, respecitvely, below the main diagonal. Similarly, we want two superdiagonals with a and b placed $M+1$ and $M-1$, respectively, above the main diagonal. The values in the vector $\underline{b}$ will mostly be zero, except for those points connecting to the boundary, where the solution is given.

To illustrate $\underline{A}$ and $\underline{b}$, let $M = N = 3$:

$$\underline{A} = \begin{pmatrix} e & & & & a & & & & \\ & e & & b & & a & & & \\ & & e & & b & & & & \\ & c & & e & & & & a & \\ d & & c & & e & & b & & a \\ & d & & & & e & & b & \\ & & & & c & & e & & \\ & & & d & & c & & e & \\ & & & & d & & & & e \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} -bg_{0,1} - cg_{1,0} - dg_{0,0} \\ -cg_{3,0} - dg_{1,0} \\ -ag_{4,2} - cg_{4,0} - dg_{2,0} \\ -bg_{0,3} - dg_{0,1} \\ 0 \\ -ag_{4,3} - cg_{4,1} \\ -ag_{2,4} - bg_{0,4} - dg_{0,2} \\ -ag_{3,4} - bg_{1,4} \\ -ag_{4,4} - bg_{2,4} - cg_{4,2} \end{pmatrix}.$$

So the matrix is block diagonal. (If you're not convinced, partition the matrix into 3×3-blocks.)

c) We have (roughly) five elements for each row, and 1000^2 unknowns. This gives a filling rate of

$$\frac{5}{1000^2} \cdot 10^2\% = 5 \cdot 10^{-4}\%,$$

vanishingly few. Storing all the zeros would be quite wasteful.

d) The Kronecker product replaces an element in a matrix with another matrix multiplied with that element. We can exploit this when we have blockdiagonal operators (as in this case). We start with two matrices on the form (again, $M = N = 3$ for illustrative purposes)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & a & 0 \\ b & 0 & a \\ 0 & b & 0 \end{pmatrix}.$$

The Kronecker product of the first with the second yields

$$\begin{pmatrix} & & & & a & & & & \\ & & & b & & a & & & \\ & & & & b & & & & \\ & & & & & & & a & \\ & & & & & & b & & a \\ & & & & & & & b & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{pmatrix}.$$

Similarly, we can construct the lower triangluar part by taking the Kronecker product of the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & c & 0 \\ d & 0 & c \\ 0 & d & 0 \end{pmatrix}.$$

Summing these, all we need is e along the main diagonal, which is easy to construct by calling *spdiags*. The code is given below.

```
function svar = lagA(M,N,a,b,c,d,e)
% v1 = [zeros(1,M+1) repmat([a*ones(1,M-1) 0],1,N)]';
% v2 = [zeros(1,M-1) repmat([0 b*ones(1,M-1)],1,N)]';
% v3 = [repmat([0 c*ones(1,M-1)],1,N) zeros(1,M-1)]';
% v4 = [repmat([d*ones(1,M-1) 0],1,N) zeros(1,M+1)]';

% svar = e*speye(M*N)...
% +spdiags(v1,M+1,M*N,M*N)...
% +spdiags(v2,M-1,M*N,M*N)...
% +spdiags(v3,-(M-1),M*N,M*N)...
% +spdiags(v4,-(M+1),M*N,M*N);

% solution with kron
B1 = spdiags([d*ones(M,1) c*ones(M,1)],[-1 1],M,M);
B2 = spdiags([b*ones(M,1) a*ones(M,1)],[-1 1],M,M);
C1 = spdiags(ones(N,1),1,N,N);
C2 = spdiags(ones(N,1),-1,N,N);

svar = kron(C1,B1)+kron(C2,B2)+speye(N*M)*e;
```

e) The matrix structure is mostly the same, but somewhat more complicated to contruct due to the varying positions of the diagonals. The filling rate is not much better. We use 1001 instead of 1000, since it is odd.

$$\frac{5}{500 \cdot 1001 + 501 \cdot 500} = 6.67 \cdot 10^{-4}\%$$

```
function svar = lagAL(M,a,b,c,d,e)
N1 = ((M-1)/2)*M; % number of unknowns in the lower half
N2 = ((M+1)/2)*(M-1)/2; % number of unknowns in the upper quarter
N = N1+N2; % total number of unknowns

v1 = [zeros(1,M+1) repmat([a*ones(1,M-1) 0],1,N1/M-1) zeros(1,N2-(M+1)+M)]'  % lower half ne
v2 = [zeros(1,M-1) repmat([0 b*ones(1,M-1)],1,N1/M-1) zeros(1,N2-(M-1)+M)]'; % lower half nw
v3 = [repmat([0 c*ones(1,M-1)],1,N1/M) zeros(1,N2)]'; % lower half se
v4 = [repmat([d*ones(1,M-1) 0],1,N1/M) zeros(1,N2)]'; % lower half sw

v5 = [zeros(1,N1) repmat([a*ones(1,(M-1)/2-1) 0],1,(M+1)/2-1) zeros(1,(M-1)/2)]'; % upper quarter ne
v6 = [zeros(1,N1) repmat([0 b*ones(1,(M-1)/2-1)],1,(M+1)/2-1) zeros(1,(M-1)/2)]'; % upper quarter nw
v7 = [zeros(1,N1) repmat([0 c*ones(1,(M-1)/2-1)],1,(M+1)/2)]' % upper quarter se
v8 = [zeros(1,N1+(M-1)/2) repmat([d*ones(1,(M-1)/2-1) 0],1,(M+1)/2-1)]'; % lower half sw
v9 = [zeros(1,N1) d*ones(1,(M-1)/2-1) zeros(1,N2-((M+1)/2-1))]' % lower border sw

svar =  e*speye(N)...
        +spdiags(v1,M+1,N,N)...
        +spdiags(v2,M-1,N,N)...
        +spdiags(v3,-(M-1),N,N)...
        +spdiags(v4,-(M+1),N,N)...
        +spdiags(v5,(M+1)/2,N,N)...
        +spdiags(v6,(M-1)/2-1,N,N)...
        +spdiags(v7,-((M-1)/2-1),N,N)...
        +spdiags(v8,-(M+1)/2,N,N)...
        +spdiags(v9,-((M+1)/2),N,N);
```

f) We need our friend Taylor. We develop each term about $U_{m,n}$ up to second order. Superscript means derivatives.

$$\begin{aligned}
aU_{m+1,n+1} &= a\left(U_{m,n+1} + hU^{x}_{m,n+1} + \frac{h^2}{2}U^{xx}_{m,n+1}\right) \\
&= a\left(U_{m,n} + kU^{y}_{m,n} + \frac{k^2}{2}U^{yy}_{m,n}\right) \\
&+ ah\left(U^{x}_{m,n} + kU^{xy}_{m,n}\right) \\
&+ \frac{ah^2}{2}\left(U^{xx}_{m,n}\right)
\end{aligned}$$

$$\begin{aligned}
bU_{m-1,n+1} &= b\left(U_{m,n+1} - hU^{x}_{m,n+1} + \frac{h^2}{2}U^{xx}_{m,n+1}\right) \\
&= b\left(U_{m,n} + kU^{y}_{m,n} + \frac{k^2}{2}U^{yy}_{m,n}\right) \\
&- bh\left(U^{x}_{m,n} + kU^{xy}_{m,n}\right) \\
&+ \frac{bh^2}{2}\left(U^{xx}_{m,n}\right)
\end{aligned}$$

$$\begin{aligned}
cU_{m+1,n-1} &= c\left(U_{m,n-1} + hU^{x}_{m,n-1} + \frac{h^2}{2}U^{xx}_{m,n-1}\right) \\
&= c\left(U_{m,n} - kU^{y}_{m,n} + \frac{k^2}{2}U^{yy}_{m,n}\right) \\
&+ ch\left(U^{x}_{m,n} - kU^{xy}_{m,n}\right) \\
&+ \frac{ch^2}{2}\left(U^{xx}_{m,n}\right)
\end{aligned}$$

$$\begin{aligned}
dU_{m-1,n-1} &= d\left(U_{m,n-1} - hU^{x}_{m,n-1} + \frac{h^2}{2}U^{xx}_{m,n-1}\right) \\
&= d\left(U_{m,n} - kU^{y}_{m,n} + \frac{k^2}{2}U^{yy}_{m,n}\right) \\
&- dh\left(U^{x}_{m,n} - kU^{xy}_{m,n}\right) \\
&+ \frac{dh^2}{2}\left(U^{xx}_{m,n}\right)
\end{aligned}$$

Now, gathering the coefficients in front of alike derivatives and demanding them all to vanish, except for those corresponding to $U^{xx}_{m,n}$ and $U^{yy}_{m,n}$, which must be one, yields the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with solution $a = b = c = d = \frac{1}{4}, e = -1$.