TMA4212 Numerical solution of partial differential equations with finite difference methods

Solutions to Problem Set 5

Problem 1. Using the triangle inequality, and $\|Cv\| \leq \|C\|\,\|v\|$, we find

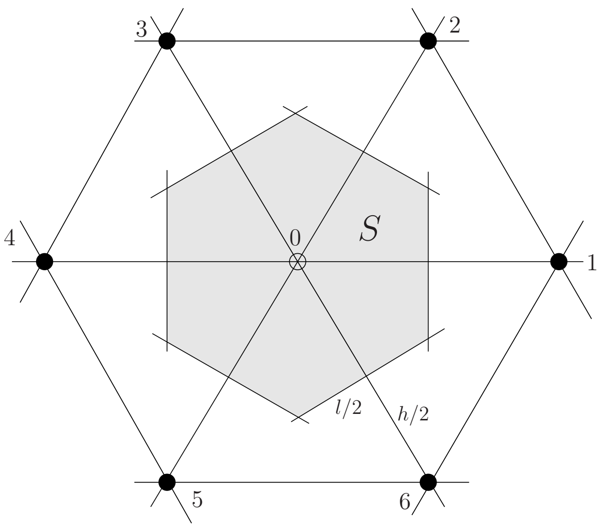
$$\begin{aligned}\|e^n\| &\leq \|v^{n-1}\| + \|C\|\|v^{n-2}\| + \cdots + \|C^{n-2}\|\|v^1\| + \|C^{n-1}\|\|v^0\| \\ &\leq nL \max_{l\leq n-1} \|v^l\| \leq TL \max_{l\leq T/k} \left\|\frac{1}{k}v^l\right\|\end{aligned}$$

since $T/k \geq n$.

Problem 2. We will find an approximation to the Laplacian on the form

$$u_{xx} + u_{yy} \approx \sum_{i=0}^{6} a_i u_i,$$

where u_i is the function value at point i (see figure). We can achieve this by studying the function's Taylor expansion at the various points, or we can use box integration. We choose the last approach, and start with Green's identity (or Gauss' theorem in 2D), giving



$$\iint_S \Delta u(x,y)\,dx\,dy = \int_{\partial S} \frac{\partial u}{\partial n}\,dl, \tag{1}$$

where S is the marked area in the figure, and the last expression above indicates the derivative in the outward normal direction. The boundary ∂S consists of six lines, each with length l, and on each of these we approximate $\partial u/\partial n$ by $(u_i - u_0)/h$, so

$$\int_{\partial S} \frac{\partial u}{\partial n}\,dl \approx \sum_{i=1}^{6} \frac{u_i - u_0}{h} l.$$

The length l can be computed using simple geometry to $l = h\tan 30^\circ = h/\sqrt{3}$, and on the other hand we have

$$\iint_S \Delta u(x,y)\,dx\,dy \approx A(S)\Delta u_0,$$

where $A(S)$ is the area of S, given by

$$A(S) = 6\frac{1}{2}(lh/2) = \frac{\sqrt{3}}{2}h^2.$$

By comparing these, we find

$$\Delta u_0 \approx \frac{2}{3h^2}\left(\sum_{i=1}^{6} u_i - 6u_0\right).$$

We are also interested in the truncation error

$$\tau = \frac{2}{3h^2}\left(\sum_{i=1}^{6} u_i - 6u_0\right) - \Delta u_0, \tag{2}$$

and we develop u_i to a suitably high order in h. Due to symmetry, all odd powers will disappear, so we may consider the even powers only. These are

$$\begin{aligned}
u_1 + u_4 &= u(x_0 + h, y_0) + u(x_0 - h, y_0) = 2u + h^2 u_{xx} + \frac{h^4}{12}u_{4x} + \mathcal{O}(h^6),\\
u_2 + u_5 &= u(x_0 + h/2, y_0 + \sqrt{3}h/2) + u(x_0 - h/2, y_0 - \sqrt{3}h/2)\\
&= 2u + h^2\left(\frac{1}{4}u_{xx} + \frac{\sqrt{3}}{2}u_{xy} + \frac{3}{4}u_{yy}\right)\\
&\quad + \frac{h^4}{12}\left(\frac{1}{16}u_{4x} + \frac{\sqrt{3}}{4}u_{3xy} + \frac{9}{8}u_{xxyy} + \frac{3\sqrt{3}}{4}u_{x3y} + \frac{9}{16}u_{4y}\right) + \mathcal{O}(h^6),\\
u_3 + u_6 &= u(x_0 - h/2, y_0 + \sqrt{3}h/2) + u(x_0 + h/2, y_0 - \sqrt{3}h/2)\\
&= 2u + h^2\left(\frac{1}{4}u_{xx} - \frac{\sqrt{3}}{2}u_{xy} + \frac{3}{4}u_{yy}\right)\\
&\quad + \frac{h^4}{12}\left(\frac{1}{16}u_{4x} - \frac{\sqrt{3}}{4}u_{3xy} + \frac{9}{8}u_{xxyy} - \frac{3\sqrt{3}}{4}u_{x3y} + \frac{9}{16}u_{4y}\right) + \mathcal{O}(h^6),
\end{aligned}$$

By insertion, we find the truncation error

$$\tau = \frac{h^2}{16}\left(u_{4x} + 2u_{2x2y} + u_{4y}\right) + \mathcal{O}(h^4).$$

All this Taylor expansion is tiring, and software like Mathematica or Maple can do it for you. In Maple, the most important command is `mtaylor`. For the above example, we can use the following:

```
xy := [seq([h*cos(k*Pi/3),h*sin(k*Pi/3)],k=0..5)];
ud := unapply( mtaylor(u(x,y),[x=0,y=0],5), x,y):
laplace:= D[1,1](u)(0,0) + D[2,2](u)(0,0):
tt := map( each_xy -> ud(op(each_xy)), xy):
ser := (2/3)*(convert(tt,'+') - nops(xy)*u(0,0))/h^2:
avbrudd := simplify(ser - laplace);
```

Problem 3. We will find a simple difference approximation to u_{xy} on a uniform rectangular grid. We try involving the points $u(x \pm h, y \pm h)$, which have Taylor expansions

$$
\begin{aligned}
u(x \pm h, y + h) &= u \pm h u_x + h u_y + \frac{h^2}{2}(u_{xx} + u_{yy}) \pm h^2 u_{xy} \\
&\qquad + \frac{h^3}{6}(\pm u_{3x} + 3u_{2xy} \pm 3u_{x2y} + u_{3y}) + \mathcal{O}(h^4), \\
u(x \pm h, y - h) &= u \pm h u_x - h u_y + \frac{h^2}{2}(u_{xx} + u_{yy}) \mp h^2 u_{xy} \\
&\qquad + \frac{h^3}{6}(\pm u_{3x} - 3u_{2xy} \pm 3u_{x2y} - u_{3y}) + \mathcal{O}(h^4).
\end{aligned}
$$

By combining these with the usual points, $u(x \pm h, y)$ og $u(x, y \pm h)$, we find for instance

$$
\frac{1}{4h^2}\left[u(x+h, y+h) - u(x-h, y+h) - u(x+h, y-h) + u(x-h, y-h)\right] = u_{xy} + \mathcal{O}(h^2),
$$

$$
\begin{aligned}
&\frac{1}{2h^2}\{u(x+h, y+h) + u(x-h, y-h) - \\
&\qquad [u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h)] + 2u(x, y)\} = u_{xy} + \mathcal{O}(h^2).
\end{aligned}
$$

Problem 4. We use Green's identity (1) on the area S given in the figure. We put the origin in P and get

$$
\begin{aligned}
\int_{\partial S} \frac{\partial u}{\partial n}\, dl &= \int_{h/2}^{-h/2} \left[-\frac{\partial u}{\partial x}(-h/2, y)\right](-dy) + \int_{-h/2}^{h/2} \left[-\frac{\partial u}{\partial y}(x, -h/2)\right] dx + \int_{\gamma'} (g - au)\, dl \\
&\approx -h u_x(-h/2, 0) - h u_y(0, -h/2) + \sqrt{2}h[g(P) - a(P)u(P)] \\
&\approx -[u(P) - u(V)] - [u(P) - u(S)] + \sqrt{2}h[g(P) - a(P)u(P)],
\end{aligned}
$$

where γ' is the piece of γ in ∂S, which has length $\sqrt{2}h$. We can also approximate

$$
\iint_S \Delta u(x, y)\, dx\, dy = \iint_S f(x, y)\, dx\, dy \approx \frac{h^2}{2} f(P),
$$

yielding

$$
u(V) + u(S) - [2 + \sqrt{2}h a(P)]u(P) = \frac{h^2}{2} f(P) - \sqrt{2}h g(P).
$$