Hyperbolic conservation laws.

An hyperbolic conservation law is a system of PDEs on the form (in one spatial dimension):

$$(\text{PDE}) \qquad u_t + (f(u))_x = 0$$

$u = [u_1, \cdots, u_\ell]^T$: u_i is a state variable
$f : \mathbb{R}^\ell \to \mathbb{R}^\ell$: flux function.

Examples:

Burgers equation : $u_t + \frac{1}{2}(u^2)_x = 0$

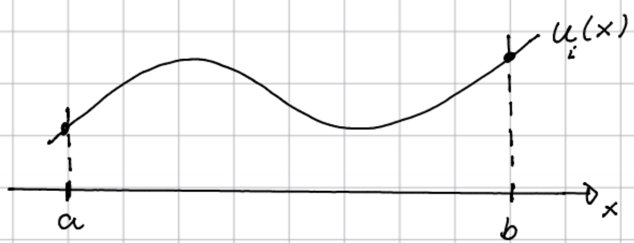
Shallow water eq : $\begin{bmatrix} v \\ \varphi \end{bmatrix}_t + \begin{bmatrix} \frac{1}{2}v^2 + \varphi \\ v\varphi \end{bmatrix}_x = 0$

(PDE) can be written in a semilinear form:

$$u_t + A(u)u_x = 0$$

with $A(u) = f_u(u)$, the jacobian of f.
So (PDE) is hyperbolic if A has a full set of eigenvectors and real eigenvalues.

Conservative:



The total amount of u_i between a and b is given by

$$\int_a^b u_i(x,t)\,dx$$

and the change in the total amount is

$$\frac{d}{dt}\int_a^b u_i(x,t)\,dx = \int_a^b \frac{d}{dt} u_i(x,t)\,dx = -\int_a^b (f_i(u))_x\,dx = f_i(a) - f_i(b).$$

The change of the total amount of u_i depends only on the flux f_i in and out of the domain.

In what follows, we discuss the scalar cons. law, hoping that the extension to systems are clear.

Conservative schemes:

Let $x_{m+1} = x_m + h$, $t_{n+1} = t_n + k$, $U_m^n \approx u(x_m, t_n)$.

A one-step numerical scheme is conservative if there is a numerical flux function F such that

$$\boxed{U_m^{n+1} = U_m^n - \rho\left[F(U_m^n, U_{m+1}^n) - F(U_{m-1}^n, U_m^n)\right], \quad \rho = \frac{k}{h}}$$

Let $T(U^n) = h\cdot\left[\frac{1}{2}U_0^n + \sum_{m=1}^{M-1} U_m^n + \frac{1}{2}U_M^n\right] \approx \int_a^b u(x,t_n)\,dx$.

The discrete version of $\frac{d}{dt}\int_a^b u\,dx = f(a) - f(b)$ is

$$(*) \quad \frac{1}{k}\left(T(U^{n+1}) - T(U^n)\right) = \frac{h}{2k}\left(U_0^{n+1} - U_0^n + U_M^{n+1} - U_M^n\right] + \frac{1}{h}\left[F(U_0^n, U_1^n) - F(U_{M-1}^n, U_M^n)\right]$$

So again, the change in the total amount only depend on what happens at the boundaries.

Partial proof of $(*)$: Let $F_m^n = F(U_m^n, U_{m+1}^n)$

$T(U^{n+1}) = h\left[\frac{1}{2}U_0^{n+1} + \frac{1}{2}U_M^{n+1} + \sum_{m=1}^{M} U_m^{n+1}\right]$ and

$$\sum_{m=1}^{M-1} U_m^{n+1} = \sum_{m=1}^{M-1} U_m^n - \rho\sum_{m=1}^{M-1}(F_m^n - F_{m-1}^n) = \sum_{m=1}^{M-1} U_m^n + \rho(F_0^n - F_{M-1}^n)$$

$$= \frac{1}{2}U_0^n + \sum U_m^n + \frac{1}{2}U_M^n - \frac{1}{2}U_0^n + \frac{1}{2}U_M^n + \rho(F_0^n - F_{M-1}^n)$$

$$= \frac{1}{h}T(U^n) - \frac{1}{2}U_0^n + \frac{1}{2}U_M^n + \rho(F_0^n - F_{M-1}^n).$$

Some examples of conservative schemes:

Lax-Friedrichs:

(PDE): $u_t + f(u)_x = 0$

(LF): $U_m^{n+1} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) - \frac{\rho}{2}(f(U_{m+1}^n) - f(U_{m-1}^n))$

This is conservative with

$$F(U_m^n, U_{m+1}^n) = \frac{1}{2}\left[\frac{1}{\rho}(U_{m-1}^n - U_m^n) + (f(U_{m+1}^n) - f(U_m^n))\right]$$

Lax-Wendroff:

For the transport equation $u_t + a u_x = 0$
Lax-Wendroff's method is

$$U_m^{n+1} = U_m^n - \frac{\rho}{2} a (U_{m+1}^n - U_{m-1}^n) + \frac{1}{2}\rho^2 a^2 \delta_x^2 U_m^n$$

For the PDE

$$u_t + f(u)_x = 0 \quad \Rightarrow \quad u_t + A(u) u_x = 0 \quad , \quad A(u) = f_u$$

the natural generalization is

$$\begin{aligned} U_m^{n+1} = U_m^n &- \frac{\rho}{2}(f(U_{m+1}^n) - f(U_{m-1}^n)) \\ &+ \frac{1}{2}\rho^2\Big[A_{m+1/2}^n (f(U_{m+1}^n) - f(U_m^n)) \\ &\qquad - A_{m-1/2}^n (f(U_m^n) - f(U_{m-1}^n))\Big] \end{aligned}$$

where $A_{m\pm 1/2}^n = A(U_{m\pm 1/2}^n)$, $\quad U_{m\pm 1/2} = \frac{1}{2}(U_m^n + U_{m\pm 1}^n)$

or $\quad A_{m\pm 1/2}^n = \frac{1}{2}(A(U_m^n) + A(U_{m\pm 1}^n))$

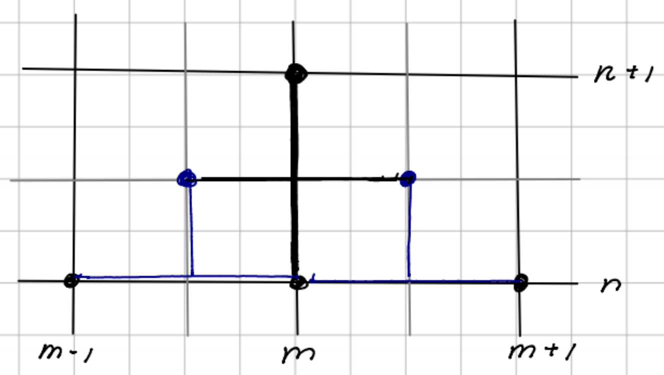
This is conservative with

$$F(U_m^n, U_{m+1}^n) = \frac{1}{2}\left(f(U_{m+1}^n) - f(U_m^n)\right)$$
$$- \frac{1}{2} p A_{m+1/2}^n \left(f(U_{m+1}^n) - f(U_m^n)\right)$$

It is of second order, but do require the evaluation of in each gridpoint,

Richtmyer-Lax-Wendroff:

This is a combination of
½ step of Lax-Friedrichs, and
½ step of Leapfrog:



$$U_{m+1/2}^{n+1/2} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) - \frac{p}{2}\left(f(U_{m+1}^n) - f(U_m^n)\right)$$
$$U_{m+1}^n = U_m^n - \frac{p}{2}\left[f(U_{m+1/2}^{m+1/2}) - f(U_{m-1/2}^{n+1/2})\right]$$

What is the numerical flux here?