Proof of Lax-Milgram theorem, given in the note by Curry, thm. 3.7.

I am going to refer to the note by Curry (CC) and the TMA4145 note by Luef (FL)

Given that all the assumptions of the theorem hold.

Choose some $u \in V$, and let

$$\varphi_u(v) = a(u, v)$$ (see FL, def. 4.3.4)

Then, for a given u, by continuity of a, $\varphi_u \in V'$, the dual space of V

By assumptions, $F \in V'$.

Riesz' representation theorem: (LF, thm. 5.8)
For each $\varphi \in V'$ there is a unique $z \in V$ s.t. $\varphi(v) = \langle z, v \rangle$, $\forall v \in V$.
And, we have $\|z\|_V = \|\varphi\|_{V'}$.

Let $\tau : V' \to V$ be the (unique and linear) operator, given by $\tau(\varphi) = z$.

Now we have three equivalent expression for our problem:

Find $u \in V$ s.t.:

$$\varphi_u(v) = F(v), \ \forall v \in V$$
$$\Updownarrow$$
$$\varphi_u = F \quad (\text{in } V')$$
$$\Updownarrow$$
$$\tau(\varphi_u) = \tau(F) \quad (\text{in } V)$$

Let $\rho \in \mathbb{R}$, $\rho \neq 0$, and define $T : V \to V$ by

$$T(w) = w - \rho \cdot (\tau(\varphi_w) - \tau(F))$$

If we can find a ρ, such that T is a contraction, then, by Banach's fixed point theorem T has a unique fixed point u; that is: (FL, thm. 3.21)

$$T(u) = u \quad \Rightarrow \quad \tau(\varphi_u) - \tau(F) = 0$$

which is then also the unique solution of the original problem.

T is a contraction if (FL, def. 3.4.3)

$$\|T(w_1) - T(w_2)\|_V < \|w_1 - w_2\|.$$

for all $w_1, w_2 \in V$.
T is linear, so by using $w = w_1 - w_2 \in V$
we only have to prove that

$$\|T(w)\|_V < \|w\|_V, \quad \forall w \in V$$

for some ρ. We have

$$\|Tw\|_V^2 = \|w - \rho\tau(\varphi_w)\|_V^2 \qquad \text{(Inner product norm)}$$

$$= \|w\|_V^2 - 2\rho\langle w, \tau(\varphi_w)\rangle_V + \rho^2\|\tau(\varphi_w)\|_V^2$$

By the definition of τ and φ_w:
$\langle w, \tau(\varphi_w)\rangle_V = \varphi_w(w) = a(w,w)$
$\|\tau(\varphi_w)\|_V^2 = \varphi_w(\tau(\varphi_w)) = a(w, \tau(\varphi_w))$
so

$$\|T(w)\|_V^2 = \|w\|_V^2 - 2\rho a(w,w) + \rho^2 a(w, \tau(\varphi_w)) \qquad \text{(Coercivity and continuity of } a\text{)}$$

$$\leq \|w\|_V^2 - 2\rho\alpha \cdot \|w\|_V^2 + \rho^2 M\|w\|_V \cdot \|\tau(\varphi_w)\|_V.$$

$$\leq (1 - 2\rho\alpha + \rho^2 M^2)\|w\|_V^2$$

The last inequality holds since by Riesz and the definition of τ:

$$\|\tau(\varphi_w)\|_V = \|\varphi_w\|_{V'} = \sup_{\substack{v \in V \\ \|v\|_V = 1}} |\varphi_w(v)|$$

and

$$|\varphi_w(v)| = |a(w,v)| \leq M \cdot \|w\|_V \cdot \|v\|_V \quad \forall w, v \in V$$

So, T is a contraction on V if

$$|1 - 2\rho\alpha + \rho^2 M^2| < 1$$

which is the case if $\rho \in (0, \frac{2\alpha}{M^2})$.

In conclusion, there is a contraction T on V with u as the unique fixed point, thus, the variational problem has a unique solution.

$\square$