TMA4212 Num.diff.
Spring 2020

Project 1

Norwegian University of Science and Technology
Department of Mathematical Sciences

Practical information

- *Deadline and hand-in:* Monday February 24 (before midnight). Hand in the project in Blackboard.
- *Supervision:* No supervision in week 6. In week 7-8, there will be some additional meeting hours, these will be announced on the wiki-page.

  In week 8, each group should have a meeting with Anne, this is considered as mandatory. At this point, you have probably not finished the work, but you should have started. In this meeting you should present what you have done so far and what you plan to do for the rest of the project. If you have met obstacles on the road, describe them as specific as possible. Meet prepared to this meeting.

  We will make a doodle form (or something similar) to schedule these meetings, and this will be available some time in week 7.
- *Report:* The report can be written as a pdf-document, with the python code in a separate file, or as Jupyter file. Write the report as a scientific report, not as a solution to an exercise. Meaning: Describe the problem you want to solve, describe the method you are using, write mathematical results as mathematical statements, and make sure there is a consistency between theoretical results and numerical etc. Use plots whenever appropriate, make sure they are readable, and explain clearly what you observe, and if it is as expected.

  The tex-report should not exceed 10 pages, and all included.
- *Grading:* Out of 20 points, the report counts for 5 points, Problem 1 for 10 and Problem 2 for 5. Roughly.
- *Learning objectives:* When completed this project you should demonstrate that you are able to:
  - develop and implement a finite difference scheme for linear and nonlinear elliptic problems.
  - perform an error analysis.
  - choose good test problems for verification of theoretical results.
  - identify and solve potential deficiencies of a scheme.
  - communicate the results in a scientific manner.

Some advice:

- *Implementation:* Make a plan. Do not to implement everything at once, split the work in small pieces, and make sure each of them works before you continue. If possible, use nontrivial test problems of which the numerical solution is exact to check that the implementation is correct, but please do not include such results in the report. You are of course allowed to use ideas from the project in TMA4215.
- *Writing:* Imagine you are writing to a fellow student, who do not know about this project. How you make him/her understand and be interested in what you have done and learned during the project?

  Writing takes a lot of time, so start early. And accept that you may want to rewrite parts, that is a part of the writing process.
- *Time organisation:* Think about how much time you are willing to use on this project. If you are completely stuck at one point, maybe it is better to skip it and concentrate on writing a good report instead.

$\boxed{1}$ In this problem, you will consider the diffusion-advection equation given by

$$-\mu \Delta u + \mathbf{v} \cdot \nabla u = f \text{ in } \Omega \tag{1}$$

with some appropriate boundary conditions. Here the diffusion constant $\mu > 0$, and the velocity field $\mathbf{v} : \Omega \to \mathbb{R}^2$ and the source function $f : \Omega \to \mathbb{R}$ may depend on the position. Here $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator, and ∇ is the gradient. If nothing else is specified, use $\mu = 1$.

a) Solve the problem on a unit square with Dirichlet conditions. Set up a finite difference scheme using central differences for both the second and the first derivatives of u. Implement the scheme. Write your program so it can solve the problem for different choices of $\mathbf{v}$, f and boundary conditions.

b) Do an error analysis of the scheme in Part 1, and find a bound for the global error in the gridpoints. Verify the results numerically.

c) Solve (1) on the domain Ω bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = 1$ (the part of the unit circle in the first quadrant). Choose boundary conditions yourself. Measure the order of the scheme, and explain the result (but a full error analysis is not required).

d) Let again Ω be the unit square, and set $\mu = 10^{-2}$, let $\mathbf{v} = [y, -x]$, $f = 1$ and $u = 0$ on $\partial\Omega$.

- What is the maximum step size allowed before the solution breaks down?
- You should notice that in this case, there is a very sharp front towards parts of the boundary. Try using Neumann conditions $\partial_n u = 0$ on these parts. Does it help?
- Try to find an explanation for the stability problem you observed in the first bullet point. Try to modify the scheme in order to find a remedy allowing you to take longer step sizes (you are allowed to use approximations of lower order if it helps.)

$\boxed{2}$ The following example is a model of a microelectromechanical device, described in [1]. The device consists of two surfaces of which one is a rigid metal plate, and the other elastic membrane fixed only at the boundaries.

Assuming that the rigid metal plate is located at $z = 0$, and the membrane at $z = 1$, the mathematical model is given by

$$\begin{aligned} \Delta u &= \frac{\lambda}{u^2} \qquad \text{in } \Omega, \\ u &= 1 \qquad \text{on } \partial\Omega, \end{aligned}$$

where u is the deflection of the membrane, and λ is proportional to the electrical potential working on the device.

Find a numerical approximation to the problem on a unit square, that is $\Omega = (0, 1) \times (0, 1)$. Use e.g. $\lambda = 1.5$, although you may experiment with different values. Be sure that your solution make sense from a physical point of view.

References

[1] J. A. Pelesko and T. A. Driscoll, *The effect of the small-aspect ratio approximation on canonical electrostatic MEMS models*, Journal of Engineering Mathematics, 53 (2005), 239–252.