Project 3

Norwegian University of Science and Technology
Department of Mathematical Sciences

Practical information

- *Deadline and hand-in:* Monday April 24 (before midnight). Hand in the project in ovsys.
- *Supervision:* Feel fre to contact Anne or Tale.
- *Report:* The report can be written as a pdf-document, with the python code in a separate file, or as Jupyter file. Write the report as a scientific report, not as a solution to an exercise. Meaning: Describe the problem you want to solve, describe the method you are using, write mathematical results as mathematical statements, and make sure there is a consistency between theoretical results and numerical etc. Use plots whenever appropriate, make sure they are readable, and explain clearly what you observe, and if it is as expected.

  The tex-report should not exceed 10 pages, and all included.
- *Grading:* Pass, no pass
- *Learning objectives:* When completed this project you should demonstrate that you are able to:
  - Perform a dissipation/dispersion analysis.
  - Explain observed behaviour of the numerical solution (of the transport problem)
  - Understand the effect of different boundary conditions for convective problems, and how to implement them.
  - Communicate the results in a scientific manner.

1 Given the transport problem

$$u_t + au_x = 0, \qquad 0 \le x \le 3, \qquad u(x,0) = f(x), \qquad u(0,t) = g(t). \tag{1}$$

with $a > 0$.

The following schemes will be considered.

$$\begin{aligned}
\text{FTBS} \quad & U_m^{n+1} = U_m^n - r(U_m^n - U_{m-1}^n) \\
\text{Lax-Wendroff} \quad & U_m^{n+1} = U_m^n - \frac{r}{2}(U_{m+1}^n - U_{m-1}^n) + \frac{r^2}{2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) \\
\text{Wendroff} \quad & U_m^{n+1} = U_{m-1}^n - \frac{1-r}{1+r}(U_{m-1}^{n+1} - U_m^n)
\end{aligned}$$

Here $r = ak/h$ is the Courant number, and k and h are the stepsizes in respectively the $t-$ and $x-$directions. And FTBS means *Forward in time, backward in space.*

a) First, let us see how well the methods deals with a discontinuous solution. Consider the transport problem (1), with initial and boundary values given by

$$u(x, 0) = 0, \qquad u(0, t) = 1.$$

The exact solution is

$$u(x, t) = \begin{cases} 1 & \text{for } x < at \\ 0 & \text{for } x > at, \end{cases}$$

that is a discontinuity moving to the right with speed a. Let $a = 1$, $h = 1/160$ and $k = rh$, and solve the problem for $t \leq 2$.

Try all three methods, using $r = 1$ (which should produce the exact solution), and $r = 0.5$. What do you observe in each case?

In section 7.7 in the note on difference methods, a dissipation and dispersion analysis is done for Lax-Wendroffs method. In this problem, we are interested in $|\xi|$ and the numerical wave speed α, both as a function of βh.

b) Do a dissipation/dispersion analysis of the Wendroff's method.

Plot $|\xi|$ and α as functions of βh for $\beta h \in [-\pi, \pi]$. Make similar plots for the Lax-Wendroff method.

c) The theoretical results can be confirmed by numerical experiments, as is explained e.g. in Strikwerda, 5.3 or in

Solve (1) using the initial value

$$f(x) = e^{-64(x-0.5)^2} \sin(32\pi x)$$

using the Lax-Wendroff's and Wendroff's methods with for instance $h = 1/200$, $k = 1/400$, and $a = 1$. Explain the results based on the theory above (see e.g. Strikwerda, 5.3 or in [1] on interpretations of the results). You are strongly encouraged to experiment a bit with the parameters, to get a good answer to an fundamental question: What happens if I ...?

2 Given the wave equation with a source term f:

$$u_{tt} = u_{xx} + u_{yy} + f(\mathbf{x}, t), \qquad x, y \in [-1, 1] \times [-1, 1]$$

and with initial conditions

$$u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = v_0(x, y)$$

For the wave equation, a lot of different boundary value conditions can be considered. In this exercise, you will focus on two of them:

- *Reflecting boundary:* In this case a wave hitting the boundary will be reflected back into the domain. This is represented by an homogeneous Neumann condition:

$$\frac{\partial u}{\partial \mathbf{n}} = 0.$$

- *Absorbing boundary:* Here, a wave hitting the boundary will behave as if it just passes through it. This condition is more complicated to described, and in this exercise, you are allowed to use a first order approximation suggested in [2], which for the boundary $x = -1$ is simply

$$u_t - u_x = 0.$$

For the numerical solution, use central differences in both time and space, leading to an explicit 2-step method. Make sure that the stepsizes you are using will result in a stable scheme.

The challenges are:

- How to do the first step in time?
- How to implement the boundary conditions?
  Implement both reflecting and absorbing conditions, but keep the same condition over the whole boundary.

Problem to solve:
Let $u_0 = v_0 = 0$, and

$$f(x,y,t) = \begin{cases} e^{-\alpha((x-\hat{x})^2+(y-\hat{y})^2)}\sin(\omega t), & 0 \leq t \leq \hat{t} \\ 0 & t > \hat{t} \end{cases}$$

In the beginning, it may be an idea to let $\hat{x} = \hat{y} = 0$, $\alpha = 100$ and $\hat{t} = \pi/\omega$, creating just one wave moving from the center towards the boundaries. Notice how the wave is either reflected or absorbed.

When the code seems to work, it is time to play with different configurations. Here are some suggestions:

In the case of reflecting boundaries, when integrating for sufficiently long time, notice how the waves will form a pattern. Then, try to move the source a bit, and see how this pattern will change.

Next, let $\hat{t} = \infty$ (and consider absorbing boundaries), and observe the behaviour of the waves. Then place more than one source (think of them as loudspeakers) at different places in the domain, and observe how the waves interfere.

References

[1] L. N. Trefethen, *Group velocity in finite difference schemes*, SIAM Review, 24 (2), 1982, pp. 113-136.

[2] B. Engquist and A. Majda, *Absorbing boundary conditions for numerical simulation of waves*, Proc. Natl. Acad. Sci. USA, 74 (5), 1977, pp. 1765-1766.