TMA4212 Num.diff.
Spring 2020

Project 3

Norwegian University of Science and Technology
Department of Mathematical Sciences

Practical information.

This project is individual, and you have to pass it to pass the course!

The exercises below are taken from central parts of the curriculum, which you should master to pass the course. So, please read (or at least skim through) the curriculum and the first exercises before trying to do the exercises. You are allowed to use all kind of aids, you can discuss with each other, or ask one of the teaching staff for help. You are *not* allowed to copy each others work directly, you have to make it your own.

- *Deadline:* May 8. The deadline is sharp, meaning you need a doctor's certificate or something similar to get an extension. Hand in your report in `ovsys`.
- *Supervision:* Just contact Anne or Tale.
- *Report:* Hand it in as one pdf-document. Scanned, handwritten notes are fine, as long as they are well structured and your handwriting is legible. Whenever you use theorems, lemmas, definitions, etc. from the curriculum, give a precise reference to it (page, equation number or something similar). Everything you write should be self-explanatory, and all results justified, but you can skip trivial calculations. There is no upper page limit.
- *Grading:* Pass, no pass.
  To pass, at least 80% of the project should be satisfactorily answered. So aim for 100%, allowing for some slack.
- *Learning objectives:* When finishing this project, you should master most of the central parts of the curriculum.

$\boxed{1}$ Given Poissons equation

$$u_{xx} + u_{yy} = f(x, y), \qquad (x, y) \in \Omega$$

where Ω is the domain given by

$$x \geq 0, \quad y \geq 0 \text{ and } y \leq 1 - x^2.$$

and with boundary conditions

$$u(0, y) = u(x, 0) = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \text{ at the boundary } y = 1 - x^2.$$

where n is the unit normal vector

Using constant stepsizes $h = 1/M$ in both directions, set up a numerically scheme for the equation. In particular, explain how to deal with the solutions at the boundary $y = 1 - x^2$.

In the two last exercises, k refer to stepsize in time, h the stepsize in space.

$\boxed{2}$ Given the transport equation

$$u_t - a u_x = 0, \qquad 0 \leq x \leq 1, \qquad a > 0 \tag{1}$$

with initial and boundary values

$$u(x, 0) = f(x), \qquad u(1, t) = 0.$$

The equation (1) is to be solved by Lax-Wendroff's method. To find approximations to the solution at the boundary $x = 0$, use a simple first order approximation.

a) Set up the complete scheme.
What is the CFL-condition in this case?

b) Changed: Perform a consistency and a von Neumann stability analysis of the Lax-Wendroff scheme.

c) Suggest a better way to find the boundary values at $x = 0$.

$\boxed{3}$ Given the following scheme

$$U_m^{n+1} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) - a\frac{k}{2h}(U_{m+1}^n - U_{m-1}^n)$$

for solving the transport equation $u_t + a u_x = 0$.

Perform a dispersion/dissipation analysis of the method, and explain the consequences of the results.

$\boxed{4}$ Consider the two point boundary value problem

$$-u_{xx} + cu = f, \qquad u(0) = u(1) = 0, \tag{2}$$

where c is some real constant.

The weak formulation of the problem is

$$\text{Find } u \in V \text{ such that } a(u, v) = F(v), \text{ for all } v \in V. \tag{3}$$

a) Set up the weak formulation of (2), that is, find a, F and V.
Under which conditions on c and f do you know that there is a unique solution of the weak formulation?

b) Let $V_h = \text{span}\{\varphi_i\}_{i=1}^N$ be some n-dimensional subspace of V. Explain the idea of Galerkins method. In particular, as a part of the process, you will have to solve a linear system of equations of the form

$$AU = b.$$

What is A and b in this case, and when the equation is solved, what is then expression for the numerical approximation to (3)?

Given that the conditions for an exact solution of (3) is satisfied.

What do you know about the properties of the matrix A?

Can the linear system be solved by a conjugate gradient method?

c) For the case $c = 0$, $f(x) = x$, let

$$V_h = \text{span}\{\sin(\pi x), \sin(2\pi x), \sin(3\pi x)\}.$$

What is the numerical approximation to $u(x)$ in this case?