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SOLUTION TO THE EXAM IN NUMERICAL MATHEMATICS (TMA4215)

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Problem 1 Let $f(x) = x^2 \ln x$, so that

$$S(0.25) = \frac{0.25}{3}\left(f(1.0) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2.0)\right) = 1.070594.$$

The error is given by

$$\int_1^2 x^2 \ln x \, dx - S(h) = -\frac{(b-a)}{180} h^4 f^{(4)}(\xi), \qquad \xi \in (1,2).$$

So, since $f^{(4)}(x) = -2/x^2$, we get $|f^{(4)}(\xi)| \leq 2$, and the error bound becomes

$$\left| \int_1^2 x^2 \ln x \, dx - S(0.25) \right| \leq 4.34 \cdot 10^{-5}.$$

Problem 2

a) By straightforward computations we get

$$\begin{aligned} \mathbf{x}^{(1)} &= [1.0, 1.0, 1.0, 1.0]^T, \\ \mathbf{x}^{(2)} &= [0.866670.93333, 0.92500, 0.93333]^T. \end{aligned}$$

b) The iterative scheme can be rewritten as

$$\mathbf{x}^{(k+1} = G\mathbf{x}^{(k)} + \mathbf{c}, \qquad G = N^{-1}P, \quad \mathbf{c} = N^{-1}\mathbf{b}$$

so that

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}\| \leq \|G\| \|\mathbf{x}^{(k)} - \mathbf{x}\|.$$

Now

$$N^{-1} = \begin{bmatrix} 1 & -1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} \qquad \text{so that} \qquad G = \begin{bmatrix} 1/30 & 0 & -1/15 & -1/10 \\ -1/30 & 0 & -1/30 & 0 \\ -1/40 & -1/40 & 0 & -1/40 \\ -1/30 & 0 & -1/30 & 0 \end{bmatrix}$$

and $\|G\|_\infty = 0.2$, proving the result in max-norm.

We also have the result

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq \frac{\|G\|_\infty}{1 - \|G\|_\infty} \|\mathbf{x}^{(k)} - \mathbf{x}^{(\mathbf{k-1})}\|_\infty \leq \frac{\|G\|_\infty^k}{1 - \|G\|_\infty} \|\mathbf{x}^{(1)} - \mathbf{x}^{(\mathbf{0})}\|_\infty,$$

so $\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty \leq 0.03333$. If you use the last expression, the bound will be 0.05.

Problem 3

a) We get

$$p(x) = -\frac{15}{8}x^2 + \frac{19}{8}x.$$

b) The polynomial $q(x)$ has to be of the form

$$q(x) = p(x) + \alpha\, x\,(x - 0.6)\,(x - 1).$$

The condition $q'(0.6) = 0.5$ gives $\alpha = -25/16$, so that

$$q(x) = -\frac{25}{16}x^3 + \frac{5}{8}x^2 + \frac{23}{16}x.$$

c) The expression is obviously correct for the nodes $x = 0$, 0.6 and $x = 1$. Following the idea of the proof for interpolation errors of the ordinary interpolation polynomial, choose an x different from the nodes, keep it fixed, and define a function $\phi(t)$ by

$$\phi(t) = f(t) - p(t) - \lambda w(t), \qquad \lambda = \frac{f(x) - q(x)}{w(x)},$$

and $w(x)$ is given in the exam set. $\phi(t) = 0$ at 4 distinct points, the nodes and x. But $\phi'(t)$ has at least 4 zeros as well, 3 in the intervals between the zeros of $\phi(t)$, and one in $t = 0.6$. So, by repeated use of Rolle's theorem, we get that $\phi^{(4)}(t) = 0$ at least once in the interval $(0, 1)$, this is the point called ξ_x. So for an arbitrary x we get

$$0 = \phi^{(4)}(\xi_x) = f^{(4)}(\xi_x) - 0 - \frac{f(x) - q(x)}{w(x)} \cdot 4!$$

which gives the expected result. The error expression is valid as long as $f \in C^4[0, 1]$.

Problem 4

a) Using $v = y,\ w = y'$ the first order system is

$$\begin{aligned} v' &= w & v(0) &= y_0, \\ w' &= f(t, v), & w(0) &= y'(0). \end{aligned}$$

Two step with Eulers method becomes

$$\begin{aligned} v_n &= v_{n-1} + hw_{n-1} & v_{n+1} &= v_n + hw_n \\ w_n &= w_{n-1} + hf(t_{n-1}, v_{n-1}) & w_{n+1} &= w_n + hf(t_n, v_n) \end{aligned}$$

The upper right expression for v_{n+1} can then be written as

$$v_{n+1} = v_n + \overbrace{(v_n - v_{n-1})}^{hw_{n-1}} + h^2 f(t_{n-1}, y_{n-1}).$$

Using $v_n = y_n$, we get the expected result.

In this case the local truncation error is given by

$$h^2 \tau_{n+1}(h) = y(t_{n+1}) - 2y(t_n) + y(t_{n-1}) - h^2 y''(t_{n-1}).$$

The Taylor expansion around t_{n-1} (you may choose another point) becomes

$$\begin{aligned} h^2 \tau_{n+1}(h) &= (1 - 2 + 1)y(t_{n-1}) + h(2 - 2)y'(t_{n-1}) + \frac{h^2}{2}(4 - 2)y''(t_{n-1}) \\ &+ \frac{h^3}{6}(8 - 2)y'''(t_{n-1}) + \cdots - h^2 y''(t_{n-1}) \\ &= h^3 y'''(t_{n-1}) + \cdots . \end{aligned}$$

The method is consistent of order 1. The characteristic polynomial becomes $\rho(r) = r^2 - 2r + 1 = (r-1)^2$, so it has a double root at $r - 1$, the stability condition is then satisfied.

The method is convergent of order 1.

b) Follow the same idea as when developing the Adams-Bashforth methods. The Newton backward polynomial form is

$$p(t) = p(t_n + sh) = f_n + s\nabla f_n + \frac{s(s+1)}{2}\nabla^2 f,$$

with $\nabla f_n = f_n - f_{n-1}$ and $\nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2}$. From the formula in the appendix, using $f(t, y(t)) \approx p(t)$ we get

$$\begin{aligned} y(t_{n+1}) - 2y_{t_n} + y(t_{n-1}) &\approx \int_{t_n}^{t_{n+1}} (t_n + h - t)p(t)dt - \int_{t_{n-1}}^{t_n} (t_n - h + t)p(t)dt \\ &= h^2 \left(\int_0^1 (1-s)p(t_n + sh)ds + \int_{-1}^0 (-1-s)p(t_n + sh)ds \right) \\ &= h^2 \left(\sigma_0 f_n + \sigma_1 \nabla f_n + \sigma_2 \nabla^2 f_n \right), \end{aligned}$$

with

$$\sigma_0 = \int_0^1 (1-s)ds + \int_{-1}^0 (1+s)ds = 1, \qquad \sigma_1 = \int_0^1 s(1-s)ds + \int_{-1}^0 s(1+s)ds = 0,$$

and

$$\sigma_2 = \int_0^1 \frac{s(s+1)}{2}(1-s)ds + \int_{-1}^0 \frac{s(s+1)}{2}(1+s)ds = \frac{1}{12}.$$

So, our method becomes

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left(f_n + \frac{1}{12}\nabla^2 f_n\right) = h^2\left(\frac{13}{12}f(t_n, y_n) - \frac{1}{6}f(t_{n-1}, y_{n-1}) + \frac{1}{12}f(t_{n-2}, y_{n-2})\right)$$

The stability properties of the method is as in **a)**, and the local truncation error becomes

$$h^2\tau_{n+1}(h) = \frac{1}{12}h^5 y^{(5)}(t_{n-2})) + \cdots .$$

The method is convergent of order 4.