Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam:
Name: Brynjulf Owren (93021641)

EXAM IN NUMERICAL MATHEMATICS (TMA4215)

10. December 2006
Time: 09:00–13:00, Grades due: 15.08.2006

Permitted aids: Cathegory B, all written aids permitted.
Simple calculator with empty memory allowed.

Problem 1 Calculate

$$\int_{-1}^{1} \frac{e^x \, dx}{\sqrt{1-x^2}}$$

with atleast 4 significant digits (note the given formulas).

Problem 2 The Van der Pol oscillator can be modelled by the initial value problem

$$u'' + \alpha(u^2 - 1)\, u' + u = 0, \qquad u(0) = u_0,\ u'(0) = v_0 \tag{1}$$

The method *backward Euler* (reluE) for the problem $y' = f(y)$ is given as

$$y_{m+1} = y_m + h\, f(y_{m+1}) \tag{2}$$

where h is the step size.

a) Show that (2) applied to (1) leads to the following equation to be solved (with respect to u) for $u_1 \approx u(h)$:

$$\alpha h\, u^3 - \alpha h u_0\, u^2 + (1 - \alpha h + h^2)\, u - u_0(1 - h\alpha) - h v_0 = 0 \tag{3}$$

b) Now let $\alpha = 5$, $h = 0.1$, $u_0 = 2$, $v_0 = 0$, and find u_1 with atleast 6 significant digits.

c) Find a polynomial $p(x)$ of degree at most 3 which satisfies

$$p(0) = 2,\ p'(0) = 0,\ p(h) = u^*,\ p'(h) = (u^* - 2)/h,$$

where h and u^* are arbitrary parameters. Then use these to approximate $u(0.05)$ for the Van der Pol oscillator with u_0, v_0 and α as in the previous task.

Problem 3 The tip of a robotic arm moves along a path in the xy-plane which can be described by a parabola-like graph$(x, f(x))$, that is $f(x)$ can be approximated well by a 2. degree polynomial $p(x)$. The following have been observed.

x_m	-1.0	-0.6	-0.2	0.2	0.6	1.0
y_m	`-8.7065e-03`	`3.2370e-01`	`9.4428e-01`	`1.2189`	`1.1431`	`1.0584`

You can assume that the x-values are exact, while there is some uncertainty in the y-values.

a) Find $p(x) \in \Pi_2$ which minimizes the sum of squares

$$E[p] = \sum_{m=1}^{6} (y_m - p(x_m))^2$$

b) We suspect that two of the data points are rather inexact, namely the second $(-0.6,$ `3.2370e-01`$)$, and the fourth $(0.2, 1.2189)$. We want to give these a somewhat reduced weight in the sum of square, which leads us to defining

$$E_w[p] = \sum_{m=1}^{6} w_m (y_m - p(x_m))^2.$$

We now let $w_1 = w_3 = w_5 = w_6 = 1$ and $w_2 = w_4 = \frac{1}{2}$. We can define an inner product on $\mathbf{R}^6$ by

$$\langle u, v \rangle_w = \sum_{m=1}^{6} w_m u_m v_m$$

To a polynomial $\phi \in \Pi_2$ we associate the vector $\bar{\phi} = [\phi(x_1), \ldots, \phi(x_6)]^T \in \mathbf{R}^6$. It turns out that the three polynomials

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = x - \frac{1}{25}, \quad \phi_2(x) = x^2 - \frac{13}{25}$$

satisfies $\langle \bar{\phi}_k, \bar{\phi}_\ell \rangle_w = 0$ when $k \neq \ell$. Use this (without proving it) to find $q \in \Pi_2$ which minimizes $E_w[q]$.

Some useful formulas

1. **Chebyshev-quadrature**

$$I(f) = \int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}} \approx Q_n(f) = \sum_{k=1}^{n} c_{n,k} f(x_{n,k})$$

$$\left.\begin{array}{rcl} c_{n,k} & = & \frac{\pi}{n} \\ x_{n,k} & = & \cos\frac{2k-2n-1}{2n}\pi \end{array}\right., \quad k = 1, \ldots, n.$$

2. **Error in Chebyshev-quadrature**

$$I(f) - Q_n(f) = \frac{\pi}{(2n)!\,2^{2n-1}}\, f^{(2n)}(\xi), \quad -1 < \xi < 1$$