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Contact during exam:
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

Monday December 8, 2008
Time: 09:00 – 13:00 Final Grade January 8.

Permitted aids (code B):
All printed and hand written aids.
Approved calculator

Problem 1 Find the polynomial of degree as low as possible, interpolating the points

x	-1.0	0.0	0.5	1.0
y	0.25	-0.5	0.25	2.75

Problem 2 Find an approximation to the integral

$$\int_0^1 \sqrt{x}\sin(x)dx \tag{1}$$

using Romberg integration. Find R_{33}.

We will now study the accuracy of Romberg integration. Let R be the Romberg table, I the exact value of the integral, so that $E = I - R$ is a table of the errors in each element of the Romberg table, that is $E_{kj} = I - R_{kj}$. Applied to the given integral, we get the following table of errors (in this table we have included values up to E_{55}).

5.6514e-02				
1.5648e-02	2.0266e-03			
4.2178e-03	4.0765e-04	2.9972e-04		
1.1111e-03	7.5583e-05	5.3445e-05	4.9536e-05	
2.8799e-04	1.3602e-05	9.4703e-06	8.7722e-06	8.6124e-06

Doing a second experiment, by applying Romberg integration to the integral $\int_0^1 \sin(x)dx$ will give the following error table:

3.8962e-02				
9.6172e-03	1.6450e-04			
2.3968e-03	1.0051e-05	2.4553e-07		
5.9872e-04	6.2467e-07	3.7420e-09	9.5981e-11	
1.4965e-04	3.8987e-08	5.8109e-11	3.6549e-13	9.5479e-15

Comment on the result.

Problem 3

a) Find the first 3 polynomials orthogonal to the inner product

$$\langle f, g \rangle_w = \int_0^1 \frac{1}{\sqrt{x}} f(x)g(x)dx.$$

b) Find a quadrature formula on the form

$$\int_0^1 \frac{1}{\sqrt{x}} f(x)dx \approx A_1 f(x_1) + A_2 f(x_2).$$

with an optimal degree of precision. Use this formula to find an approximation to the integral (1).

Problem 4 Modified Euler is a 2nd order Runge–Kutta method, described by the Butcher-tableaux

$$\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array} \quad . \tag{2}$$

We will use this method to solve the differential equation

$$y'' = y' \cdot y, \qquad y(0) = 1, \quad y'(0) = 0.5. \tag{3}$$

a) Reformulate (3) to a system of first order differential equations. Do one step with the method to find approximations to $y(0.1)$ and $y'(0.1)$.

b) Consider a Runge–Kutta method in 3 stages, given by

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ c_3 & a_{31} & a_{32} & \\ \hline & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \end{array} \quad .$$

Let $c_3 = 1$. What should the remaining parameters be for the method to be of order 3?

The method (2) together with the 3th order method from point **b)** form an embedded Runge–Kutta pair, which can be used to construct an adaptive algorithm for solving ordinary differential equations. Let us now use this pair to solve (3) with a tolerance of 10^{-3}.

c) From **a)** we have the solutions $y_1 \approx y(0.1)$ and $y_1' \approx y'(0.1)$.

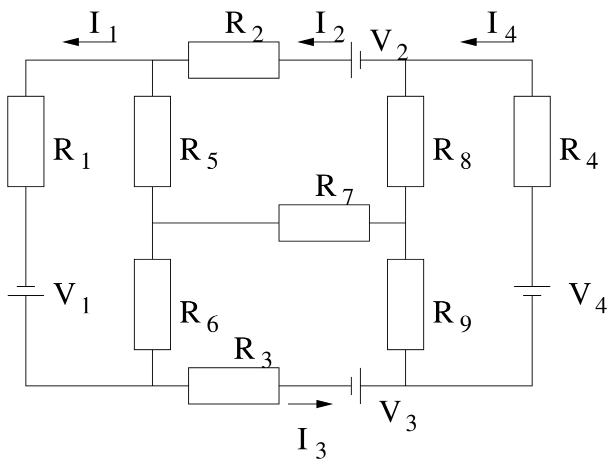
- Give an estimate for the error (use the max-norm).
- Can the solution be accepted, or should it be rejected.
- Compute the next stepsize.
  Use error pr. step (EPS) and a pessimist factor $P = 0.75$ for the stepsize control.

Hint: The order conditions for Runge–Kutta methods are given by

order	condition
1	$\sum b_i = 1$
2	$\sum b_i c_i = 1/2$
3	$\sum b_i c_i^2 = 1/3$
	$\sum b_i a_{ij} c_j = 1/6$

with $c_i = \sum_j a_{ij}$.

Problem 5 Consider the following electrical circuit.



We would like to find the currents I_i, $i = 1, \ldots, 4$ when the resistances R_i, $i = 1, \ldots, 9$ and the voltage sources V_i, $i = 1, \ldots, 4$ are given. Kirshoffs laws together with Ohm's law gives the following system of equations:

$$\begin{aligned} R_1 I_1 + R_5(I_1 - I_2) + R_6(I_1 - I_3) &= V_1, \\ R_2 I_2 + R_5(I_2 - I_1) + R_7(I_2 - I_3) + R_8(I_2 - I_4) &= V_2, \\ R_3 I_3 + R_6(I_3 - I_1) + R_7(I_3 - I_2) + R_9(I_3 - I_4) &= V_3, \\ R_4 I_4 + R_8(I_4 - I_2) + R_9(I_4 - I_3) &= V_4. \end{aligned}$$

You can assume that $R_i > 0$, $i = 1, \ldots, 4$,while $R_i \geq 0$ for $i = 5, \ldots, 9$.

a) Explain why this system of equations always will have a solution, and why Jacobi or Gauss-Seidel iterations always will converge to the solution, independent of the choice of starting values.

b) Let $R_1 = R_2 = 10$, $R_3 = R_4 = 25$, $R_i = 15$, $i = 5, \ldots, 9$, $V_1 = V_4 = 6$, $V_2 = V_3 = 0$. Do one Gauss-Seidel iteration on the system. Use $I_i^{(0)} = V_i/R_i$, $i = 1, \ldots, 4$ as starting values.