

Bernoulli polynomials and the Euler-Maclaurin formula.

Let the integral

$$I(f) = \int_a^b f(x)dx$$

be approximated by the Trapezoidal rule

$$T(h) = h \left(\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n) \right)$$

where $x_i = a + ih$, $i = 0, \dots, n$ with $h = (b - a)/n$. The aim of this note is to prove that the error can be written as

$$I(f) - T(h) = \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} h^{2k} (f^{(2k-1)}(a) - f^{(2k-1)}(b)) - \frac{b_{2m}}{(2m)!} h^{2m} (b - a) f^{(2m)}(\eta) \quad (1)$$

where $\eta \in (a, b)$, and b_k are the Bernoulli numbers. These will be defined later. Obviously, the formula requires $f \in C^{2m}[a, b]$. The formula proves that the error can be written as an even power series of h , which is the theoretical fundament for the development of the Romberg integration algorithm.

We will first define the well known Bernoulli polynomials which will then be used to prove Euler-Maclaurins formula. This again will be used to prove (1).

Bernoulli polynomials.

For our purpose, it is convenient to define the Bernoulli polynomials by the following recurrence relation:

$$B_0(t) = 1, \\ B'_k(t) = kB_{k-1}(t) \quad \text{and} \quad \int_0^1 B_k(t) dt = 0, \quad k \geq 1.$$

The first few polynomials becomes

$$B_1(t) = t - \frac{1}{2} \\ B_2(t) = t^2 - t + 1/6 \\ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

We will need the following properties of these polynomials:

$$B_k(0) = B_k(1), \quad k \geq 2, \quad (2a)$$

$$B_k(1 - t) = (-1)^k B_k(t), \quad (2b)$$

$$B_k(t) - B_k(0) \quad \text{has no zeros in } (0,1) \text{ if } k \text{ is even.} \quad (2c)$$

The *Bernoulli numbers* is given by $b_k = B_k(0)$. As a consequence of (2a) and (2b) we get $b_k = 0$ for k odd and $k \geq 3$. The Bernoulli numbers can also be found by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{b_k}{k!} t^k.$$

Euler-Maclaurins formula

By repeated use of integration by parts and the Bernoulli polynomials, we get

$$\begin{aligned}
 \int_0^1 f(t)dt &= \int_0^1 f(t)B_0(t)dt \\
 &= f(t)B_1(t)\Big|_0^1 - \int_0^1 B_1(t)f'(t)dt \\
 &= \frac{1}{2}[f(1) + f(0)] - \frac{1}{2}B_2(t)f'(t)\Big|_0^1 + \frac{1}{2}\int_0^1 B_2(t)f''(t)dt \\
 &= \frac{1}{2}[f(1) + f(0)] - \sum_{j=2}^{\bar{m}} \frac{(-1)^j}{j!} B_j(t)f^{(j-1)}(t)\Big|_0^1 + (-1)^{\bar{m}} \frac{1}{\bar{m}!} \int_0^1 B_n(t)f^{(n)}(t)dt.
 \end{aligned}$$

Now, since $B_{2k+1}(0) = B_{2k+1} = 0$ and $B_{2k}(0) = B_{2k}(1) = b_{2k}$ this can be written as

$$\begin{aligned}
 \int_0^1 f(t)dt &= \frac{1}{2}[f(1) + f(0)] - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} [f^{(2k-1)}(1) - f^{(2k-1)}(0)] \\
 &\quad - \frac{b_{2m}}{(2m)!} [f^{(2m-1)}(1) - f^{(2m-1)}(0)] + \frac{1}{(2m)!} \int_0^1 B_{2m}(t)f^{(2m)}(t)dt.
 \end{aligned}$$

Using property (2c) and the mean value theorem for integrals, the last two terms becomes

$$-\frac{1}{(2m)!} \int_0^1 [b_{2m} - B_{2m}(t)] f^{(2m)}(t)dt = -\frac{b_{2m}}{(2m)!} f^{(2m)}(\eta)$$

As a result, we get the *Euler-Maclaurin formula*:

$$\int_0^1 f(t)dt = \frac{1}{2}[f(1) + f(0)] - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} [f^{(2k-1)}(1) - f^{(2k-1)}(0)] - \frac{b_{2m}}{(2m)!} f^{(2m)}(\eta).$$

Error of the Trapezoidal rule.

Applying the Euler-Maclaurin formula to a function $g(t) = f(x_i + ht)$, using the change of variable $t = (x - x_i)/h$ gives

$$\begin{aligned}
 \int_{x_i}^{x_{i+1}} f(x)dx &= \frac{h}{2}[f(x_i) + f(x_{i+1})] - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} h^{2k} [f^{(2k-1)}(x_{i+1}) - f^{(2k-1)}(x_i)] \\
 &\quad - \frac{b_{2m}}{(2m)!} h^{2m+1} f^{(2m)}(\eta_i)
 \end{aligned}$$

The expression (1) is finally obtained by summing over all the intervals $[x_i, x_{i+1}]$.