6 Collocation.

We will now see how the idea of orthogonal polynomials can be used to construct high order Runge-Kutta methods.

Given an ordinary differential equation

$$y'(t) = f(t, y(t)), \qquad y(t_n) = y_n, \qquad t \in [t_n, t_n + h].$$

Recall also the defintion of a Runge–Kutta method applied to the ODE:

$$\begin{aligned} k_i &= f(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij} k_j) \\ y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i k_i. \end{aligned} \tag{23}$$

The idea of collocation methods for ODEs are as follows: Assume that s distinct points $c_1, c_2, \ldots, c_s \in [0, 1]$ are given. we will then search for a polynomial $u \in \mathbb{P}_s$ satisfying the ODE in the points $t_n + c_i h$, $i = 1, \cdots, s$. These are the *collocation conditions*, expressed as

$$u'(t_n + c_i h) = f(t_n + c_i h, u(t_n + c_i h)), \qquad i = 1, 2, \ldots, s, \tag{24}$$

The initial condition is $u(t_n) = y_n$. When finally $u(t)$ has been found, we will set $y_{n+1} = u(t_n + h)$.

Let k_i be some (so far unknown) approximation to $y'(t_n + c_i h)$, and let $u' = p \in \mathbb{P}_{s-1}$ be the interpolation polynomial satisfying $p(t_n + c_i h) = k_i$. For simplicity, introduce the change of variables $t = t_n + \tau h$, $\tau \in [0, 1]$. Then

$$p(t) = p(t_n + h\tau) = \sum_{j=1}^{s} k_j \ell_j(\tau), \qquad \ell_j(\tau) = \prod_{\substack{k=1 \\ k \neq j}}^{s} \frac{\tau - c_k}{c_j - c_k}.$$

The polynomial $u(t)$ is given by

$$u(\tilde{t}) = u(t_n) + \int_{t_n}^{\tilde{t}} p(t)dt = y_n + h \sum_{j=1}^{s} k_j \int_0^{\tilde{\tau}} \ell_j(\tau) d\tau, \qquad \tilde{t} \in [t_n, t_n + h].$$

The collocation condition becomes

$$u'(t_n + c_i h) = f\left(t_n + c_i h, y_n + \sum_{j=1} k_j \int_0^{c_i} \ell_j(\tau) d\tau.\right)$$

and in addition we get

$$u(t_n + h) = y_n + \sum_{j=1} k_j \int_0^1 \ell_j(\tau) d\tau.$$

But $u'(t_n + c_i h) = k_i$, and $y_{n+1} = u(t_n + h)$ so this is exactly the Runge–Kutta method (23) with coefficients

$$a_{ij} = \int_0^{c_i} \ell_j(\tau) d\tau, \quad i, j = 1, \ldots, s, \qquad\qquad b_i = \int_0^1 \ell_j(\tau) d\tau, \quad i = 1, \ldots, s. \tag{25}$$

Runge–Kutta methods constructed this way are called *collocation methods* and they are of order at least s. But a clever choice of the c_i's will give better results.

Example 6.1. *Let $P_2(x) = x^2 - 1/3$ be the second order Legendre polynomial. Let c_1 and c_2 be the zeros of $P_2(2x-1)$ (using a change of variable to transfer the interval [-1,1] to [0,1]), that is*

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \qquad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

The corresponding cardinal functions becomes

$$\ell_1(\tau) = -\sqrt{3}\left(\tau - \frac{1}{2} - \frac{\sqrt{3}}{6}\right), \qquad \ell_2(\tau) = \sqrt{3}\left(\tau - \frac{1}{2} + \frac{\sqrt{3}}{6}\right).$$

From (25) *we get a Runge–Kutta method with the following Butcher tableau:*

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

This method has order 4 *(check it yourself).*

Collocation methods using the zeros of the Legendre-polynomials $P_s(2x-1)$ for the c_i's are called Gauss-Legendre methods, also known as Kuntzmann-Butcher methods. They are of order $2s$, they are also A-stable.