Solution of systems of nonlinear equations

Given a system of nonlinear equations

$$F(x) = 0, \qquad F : \mathbb{R}^m \to \mathbb{R}^m \tag{1}$$

for which we assume that there is (at least) one solution $x^\star$. The idea is to rewrite this system into the form

$$x = G(x), \qquad G : \mathbb{R}^m \to \mathbb{R}^m. \tag{2}$$

The solution $x^\star$ of (1) should satisfy $x^\star = G(x^\star)$, and is thus called a *fixed point* of G. The iteration schemes becomes: given an initial guess $x^{(0)}$, the *fixed point iterations* becomes

$$x^{(k+1)} = G(x^{(k)}), \qquad k = 1, 2, \dots . \tag{3}$$

The following questions arise:

(i) How to find a suitable function G?
(ii) Under what conditions will the sequence $x^{(k)}$ converge to the fixed point $x^\star$?
(iii) How quickly will the sequence $x^{(k)}$ converge?

Point (ii) can be answered by Banach fixed point theorem:

Theorem 1. *Let $D \subseteq \mathbb{R}^m$ be a convex*[1] *and closed set. If*

$$G(D) \subseteq D \tag{4a}$$

and

$$\|G(y) - G(v)\| \le L\|y - v\|, \qquad \text{with } L < 1 \text{ for all } y, v \in D, \tag{4b}$$

then G has a unique fixed point in D and the fixed point iterations (3) *converges for all $x^{(0)} \in D$. Further,*

$$\|x^{(k)} - x^\star\| \le \frac{L^k}{1-L}\|x^{(1)} - x^{(0)}\|. \tag{4c}$$

Proof. The proof is based on the *Cauchy Convergence theorem*, saying that a sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to some $x^\star$ if and only if for every $\varepsilon > 0$ there is an N such that

$$\|x^{(l)} - x^{(k)}\| < \varepsilon \qquad \text{for all} \quad l, k > N. \tag{5}$$

Assumption (4a) ensures $x^{(k)} \in D$ as long as $x^{(0)} \in D$. From (3) and (4b) we get:

$$\|x^{(k+1)} - x^{(k)}\| = \|G(x^{(k)}) - G(x^{(k-1)})\| \le L\|x^{(k)} - x^{(k-1)}\| \le L^k\|x^{(1)} - x^{(0)}\|.$$

[1] D is convex if $\theta y + (1-\theta)v \in D$ for all $y, v \in D$ and $\theta \in [0, 1]$.

We can write $x^{(k+p)} - x^{(k)} = \sum_{i=1}^{p}(x^{(k+i)} - x^{(k+i-1)})$, thus

$$\begin{aligned}\|x^{(k+p)} - x^{(k)}\| &\leq \sum_{i=1}^{p}\|x^{(k+i)} - x^{(k+i-1)}\| \\ &= (L^{p-1} + L^{p-2} + 1)\|x^{(k+1)} - x^{(k)}\| \leq \frac{L^k}{1-L}\|x^{(1)} - x^{(0)}\|,\end{aligned}$$

since $L < 1$. For the same reason, the sequence satisfy (5), so the sequence converges to some $x^\star \in D$. Since the inequality is true for all $p > 0$ it is also true for $x^\star$, proving (4c).

To prove that the fixed point is unique, let $x^\star$ and $y^\star$ be two different fixed points in D. Then

$$\|x^\star - y^\star\| = \|G(x^\star) - G(y^\star)\| < \|x^\star - y^\star\|$$

which is impossible. $\square$

For a given problem, it is not necessarily straightforward to justify the two assumptions of the theorem. But it is sufficient to find some L satisfying the condition $L < 1$ in some norm to prove convergence.

Let $x = [x_1, \ldots, x_m]^T$ and $G(x) = [g_1(x), \ldots, g_m(x)]^T$. Let $y, v \in D$, and let $x(\theta) = \theta y + (1-\theta)v$ be the straight line between y and v. The mean value theorem for functions gives

$$\begin{aligned} g_i(y) - g_i(v) = g_i(x(1)) - g_i(x(0)) &= \frac{dg_i}{d\theta}(\tilde{\theta})(1-0), && \tilde{\theta} \in (0,1) \\ &= \sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(\tilde{x}_i)(y_j - v_j), && \tilde{x}_i = \tilde{\theta} y + (1-\tilde{\theta})v \end{aligned}$$

since $dx_j(\theta)/d\theta = y_j - v_j$. Then

$$|g_i(y) - g_i(v)| \leq \sum_{j=1}^{m}\left|\frac{\partial g_i}{\partial x_j}(\tilde{x}_i)\right| \cdot |y_j - v_j| \leq \left(\sum_{j=1}^{m}\left|\frac{\partial g_i}{\partial x_j}(\tilde{x}_i)\right|\right)\max_l |y_l - v_l|.$$

If we let $\bar{g}_{ij}$ be some upper bound for each of the partial derivatives, that is

$$\left|\frac{\partial g_i}{\partial x_j}(x)\right| \leq \bar{g}_{ij}, \quad \text{for all } x \in D.$$

then

$$\|g(y) - g(v)\|_\infty = \left(\max_i \sum_{j=1}^{m}\bar{g}_{ij}\right)\|y - v\|_\infty.$$

We can then conclude that (4b) is satisfied if

$$\max_i \sum_{j=1}^{m}\bar{g}_{ij} < 1.$$

Newton's method

Newton's method is a fixed point iterations for which

$$G(x^{(k)}) = x^{(k)} - J(x^{(k)})^{-1}F(x^{(k)}),$$

where the *Jacobian* is the matrix function

$$J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_m}(x) \end{pmatrix}.$$

It is possible to prove that if *i)* (1) has a solution $x^\star$, *ii)* $J(x)$ is nonsingular in some open neighbourhood around $x^\star$ and *iii)* the initial guess $x^{(0)}$ is sufficiently close to $x^\star$, the Newton iterations will converge to $x^\star$ and

$$\|x^\star - x^{(k+1)}\| \leq K\|x^\star - x^{(k)}\|^2$$

for some positive constant K. We say that the convergence is *quadratic*.

Steepest descent

Steepest descent is an algorithm that search for a (local) minimum of a given function $g : \mathbb{R}^m \to \mathbb{R}$. The idea is as follows.

a) Given some point $x \in \mathbb{R}^m$.
b) Find the direction of steepest decline of g from x (steepest descent direction)
c) Walk steady in this direction till g starts to increase again.
d) Repeat from a).

The direction of steepest descent is $-\nabla g(x)$, where the gradient ∇g is given by

$$\nabla g(x) = \left[\frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_m}(x)\right]^T.$$

And the steepest descent algorithm reads

function STEEPEST DESCENT(g, $x^{(0)}$)
  for k=0,1,2,.... **do**
    $z = -\nabla g(x^{(k)})/\|\nabla g(x^{(k)})\|$ ▷ The steepest descent direction.
    Minimize $g(x^{(k)} + \alpha z)$, giving $\alpha = \alpha^\star$.
    $x^{(k+1)} = x^{(k)} + \alpha^\star z$
  end for

end function

This algorithm will always converge to some point $x^\star$ in which $\nabla g(x^\star) = 0$, usually a local minimum, if one exist. But the convergence can be very slow.

This can be used to find solution of the nonlinear system of equations (1) by defining

$$g(x) = F(x)^T F(x) = \|F(x)\|_2^2.$$

Thus, $x^\star$ is a minimum of $g(x)$ if and only if $x^\star$ is a solution of $F(x) = 0$. In this case, we can show that

$$\nabla g(x) = 2J(x)^T F(x).$$