TMA4215 Numerical Mathematics

Autumn 2009

Exercise 1

Task 1

a) Given $f(x) = \sqrt{1+x}$. Let $x_0 = 0$, $x_1 = 0.9$, $x_2 = 0.6$ and $x_3 = 0.4$. Construct the interpolation polynomials of degree 1, 2 and 3 for approximating $f(0.45)$. Find the error in each case.

b) Use Theorem 2, Sec. 6.1 to find an error bound for the approximations to $f(0.45)$ in **a)**.

c) Use the MATLAB function `lagrange` to find the approximations to $f(0.45)$.

Since MATLAB now is running, make a plot of $p_3(x)$ and $f(x)$ for $x \in [0, 0.9]$. What happens if you expand the domain to e.g. $[-0.5, 1.5]$?

You may also try adding extra nodes.

Task 2

Check that the polynomials

$$p(x) = 5x^3 - 27x^2 + 45x - 21,$$
$$q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$$

both interpolate the points given in the table

x	1	2	3	4
$f(x)$	2	1	6	47

Why does this not contradict the uniqueness theorem (Theorem 1, Sec. 6.1)?

Task 3

Given a set of equidistant nodes, i.e. $x_k = a + kh$, $k = 0, 1, \ldots, n$, with $h = (b - a)/n$. Let $p_n(x)$ be the polynomial of degree n that intepolates a function f in the nodes. The task is about showing the error bound

$$|f(x) - p_n(x)| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1} \tag{1}$$

where $M = \max_{x \in [a,b]} |f^{(n+1)}(x)|$.

Choose an $x \in [a, b]$, and let j be such that $x_j \leq x \leq x_{j+1}$. Show the error bound

$$\prod_{k=0}^{n} |x - x_k| \leq \frac{1}{4} h^{n+1} (j+1)!(n-j)!.$$

You may draw a figure. It is useful to separate the product in three parts, $k < j$, $k = j, j+1$ and $k > j + 1$, and then find an upper bound for each of these.

Use this to show

$$\left| \prod_{k=0}^{n} (x - x_k) \right| \leq \frac{1}{4} h^{n+1} n!.$$

Finally, show (1).

Task 4

Given the function $f(x) = e^x \sin x$ on the interval $[-3, 1]$.

a) Show by induction that

$$f^{(m)}(x) = \frac{d^m}{dx^m} f(x) = 2^{m/2} e^x \sin(x + m\pi/4).$$

b) Let $p_n(x)$ be the polynomial interpolating $f(x)$ in $n+1$ equidistant nodes (including the end points). Find an upper limit for the error expressed using n. To guarantee an error less than 10^{-4}, what must n be? (Use trial and error, or calculate it using MATLAB or Maple).

c) Use MATLAB to verify the results in **b)**.

Task 5

This task should be done in MATLAB.

The gross domestic production of crude oil in Norway from 1980 to 2008 measured in standard cubic metres (Sm^3) is provided in Table 1. Find the interpolation polynomial of degree 7 for

Year	Oil production (10^6 Sm^3)
1980	30.688
1984	43.709
1988	66.882
1992	125.936
1996	177.282
2000	182.126
2004	161.064
2008	113.335

Table 1: Norwegian oil production 1980–2008 (source: Statistics Norway).

the points in the table. Use the polynomial to find an estimate of the oil production in 1990 (for comparison, the oil production that year was $96.844 \cdot 10^6$ Sm^3). How about forecasts for 2009 and 2010? What advice would you give the politicians?