TMA4215 Numerical Mathematics

Autumn 2009

Exercise 4

Task 1

Given an ordinary differential equation

$$y' = f(t, y), \qquad y(t_0) = y_0, \qquad t_0 \le t \le t_{\text{end}}. \tag{1}$$

You can assume that f satisfies the Lipschitz condition

$$\|f(t, y) - f(t, \tilde{y})\| \le L\|y - \tilde{y}\|.$$

A *one-step method* for solving this differential equation can be described by

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h), \qquad n = 0, 1, \dots, N-1, \quad h = \frac{t_{\text{end}} - t_0}{N} \tag{2}$$

Assume the following:

- The local truncation error given by
$$d_{n+1} = y(t_{n+1}) - y(t_n) - h\Phi\big(t_n, y(t_n); h\big)$$
satisfies
$$\|d_{n+1}\| \le Dh^{p+1}$$
where D is a positive constant.
- The function Φ is Lipschitz continuous, with Lipschitz constant M, i.e.
$$\|\Phi(t_n, y; h) - \Phi(t_n, \tilde{y}; h)\| \le M\|y - \tilde{y}\|. \tag{3}$$

a) Show that in this case, the global error in t_{end} satisfies

$$\|e_N\| = \|y(t_{\text{end}}) - y_N\| \le Ch^p,$$

where C is a positive constant depending on M, D and the interval $t_{\text{end}} - t_0$.

b) Assume that a two-stage explicit Runge–Kutta method given by the Butcher tableau

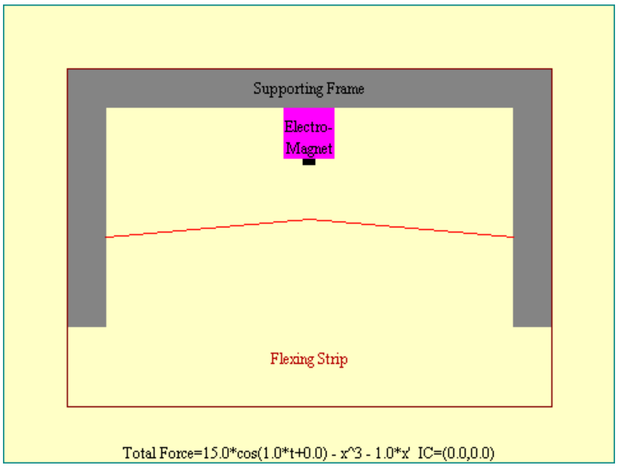
$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2 & \\ \hline & b_1 & b_2 \end{array}$$

is used to solve (1). Show that the method can be written on the form (2). Now assume that $h \le h_{\text{max}}$ and show that Φ satisfies the Lipschitz condition in y, with Lipschitz constant M that depends on the method coefficients c_2, b_1 and b_2, as well as L and h_{max}.

Task 2

The Duffing oscillator is a much studied mathematical model. This can be described by the initial value problem

$$u'' + ku' - u(1 - u^2) = A\cos(\omega t). \quad (4)$$



In 1918, G. Duffing used this equation to describe a thin, flexible metal bar oscillating near an electromagnet. The constant k is the damping, while ω and A are the frequency and the amplitude of the driving force from the electromagnet respectively. See `http://www.mcasco.com/pattr1.html` for more details.

a) Start by transforming (4) to a system of two first-order differential equations.

b) Calculate by hand (you are allowed to use a calculator) a single step with the improved Euler method, setting $k = 0.25$, $A = 0.4$, $\omega = 1.0$, $u(0) = 0$, $u'(0) = 0$, and using step length $h = 0.1$.

c) Implement the improved Euler method in MATLAB and use it to solve (4).

d) Create a plot of the first component u along the x-axis and the second component u' along the y-axis (this is called a *phase plot*). Start with the same parameters as in **b)**, but vary them and see what happens. You may use $h = 0.01$. Try integrating over quite long time intervals.

e) Try several different initial values and plot the resulting integral curves to get a picture of what the curves look like. You can use the same values as above for $k = 0.25$, $A = 0.4$, $\omega = 1.0$.

f) Finally, make an implementation where you replace improved Euler by RK4. Compare the results.

Task 3

Write down all trees of order 5 with the accompanying order conditions (there are 9 of them).