TMA4215 Numerical Mathematics

Autumn 2009

Exercise 5

Task 1

Kutta's method from 1901 is the most famous of all explicit Runge–Kutta pairs, given by the following Butcher tableau:

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}$$

a) Verify that the method has order 4 by checking all 8 order conditions.

b) An alluring thought is to now find a new set of weights, say $\hat{b}_s$ such that the accompanying method is of order 3, for error estimates and step length control. Try to find such a set of $\hat{b}_s$.

Task 2

a) Show that an explicit Runge–Kutta method with s stages maximally can be of order s. (Hint: Use $y' = y$, $y(0) = y_0$ as test equation.)

b) Show that an explicit 3rd order Runge–Kutta method with 3 stages must satisfy

$$3a_{32}c_2^2 - 2a_{32}c_2 - c_2c_3 + c_3^2 = 0.$$

c) Characterise all 3rd order explicit Runge–Kutta methods with 3 stages that satisfy $a_{31} = 0$, i.e. $a_{32} = c_3$. How many free parameters are there?

d) Find all explicit methods of order 2 that have the same coefficients a_{ij} as the method above, and weights that simultaneously satisfy $\hat{b}_3 = 0$.

Task 3

Implement an adaptive Runge–Kutta method based on the Bogacki–Shampine pair:

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
3/4 & 0 & 3/4 & & \\
1 & 2/9 & 1/3 & 4/9 & \\
\hline
 & 2/9 & 1/3 & 4/9 & 0 \\
\hline
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}
$$

Use pessimist factor $P = 0.9$.

Test your program on the Lotka–Volterra equation (see note). Plot the solution and the step lengths h_n. Does h_n vary as expected?

Relevant exam problems:

- December 2003, problem 1,
- August 2005, problem 3,
- December 2006, problem 3,
- December 2008, problem 4.