TMA4215 Numerical Mathematics

Autumn 2009

Exercise 6

Task 1

The Butcher tableau for the Bogacki–Shampine pair was provided in exercise 5. Find the stability functions of the two methods. Plot the stability region by using the MATLAB function `stab.m`.

Task 2

a) Find the eigenvalues of the matrix

$$M = \begin{pmatrix} -10 & -10 \\ 40 & -10 \end{pmatrix}.$$

b) Assume that you are to solve the differential equation

$$y' = My, \qquad y(0) = y_0$$

using the improved Euler method. What is the largest step size $h_{\max}$ you can use?

c) Solve the equation

$$y' = My + g(t), \qquad 0 \le t \le 10$$

with

$$g(t) = \big(\sin(t), \cos(t)\big)^{\mathrm{T}}, \qquad y(0) = \left(\frac{5210}{249401}, \frac{20259}{249401}\right)^{\mathrm{T}}$$

by using `impEuler.m`. Choose step sizes a little smaller than and a little larger than $h_{\max}$. What do you observe?

Task 3

a) Given a collocation method as described in the lecture notes, show that the polynomial $u(t)$ satisfying the collocation conditions

$$\begin{aligned} u'(t_n + c_i h) &= f\big(t_n + c_i h, u(t_n + c_i h)\big), \qquad i = 1, 2, \ldots, s, \\ u(t_n) &= y_n, \end{aligned}$$

can be written

$$u(t_n + \theta h) = y_n + h \sum_{i=1}^{s} b_i(\theta) k_i, \qquad \theta \in [0, 1].$$

b) Find the collocation method based on the nodes $c_1 = 1/3$ and $c_2 = 1$. Determine the method order and find the stability function.

Find $b_1(\theta)$ and $b_2(\theta)$.

A method is A-stable if $|R(z)| \leq 1$ for all $z \in \mathbb{C}^-$. The stability function R is always a rational function, i.e. $R(z) = P(z)/Q(z)$ where P and Q are polynomials in z. By using the maximum modulus principle it is possible to show that a method is A-stable if and only if

1. $R(z)$ does not have poles in $\mathbb{C}^-$ (poles are zeros of $Q(z)$),
2. $|R(y\mathrm{i})|^2 \leq 1$ for all $y \in \mathbb{R}$.

c) Show that the collocation method of **b)** is A-stable. Plot the stability region.

d) Find the system of nonlinear equations that must be solved when taking a step with this method applied to the van der Pol equation.

Task 4

Just for fun!

The linear test equation

$$y' = \lambda y, \qquad y(0) = y_0$$

has solution $y(h) = \mathrm{e}^z y_0$ where $z = \lambda h$. One step with a Runge–Kutta method gives $y_1 = R(z)y_0$. Thus, we can consider the stability function $R(z)$ as an approximation of e^z. Will $R(z)$ grow (absolutely) faster than e^z? We can find this out by studying when $|R(z)/\mathrm{e}^z| > 1$.

Rewrite the script `stab.m` so that it plots the region

$$\mathcal{A} = \{z \in \mathbb{C} \mid |R(z)/\mathrm{e}^z| > 1\}.$$

Find $\mathcal{A}$ for the stability functions you found in this exercise as well as the ones in the lecture notes. You may also draw the stability functions for the Gauss–Legendre methods (collocation methods of order $2s$). These are given by:

$$\begin{aligned} s &= 1, \qquad R(z) = \frac{1 + z/2}{1 - z/2}, \\ s &= 2, \qquad R(z) = \frac{1 + z/2 + z^2/12}{1 - z/2 + z^2/12}, \\ s &= 3, \qquad R(z) = \frac{1 + z/2 + z^2/10 + z^3/120}{1 - z/2 + z^2/10 - z^3/120}. \end{aligned}$$

The region $\mathcal{A}$ is called an *order star*.