TMA4215 Numerical Mathematics

Autumn 2009

Exercise 7

You should have read 3.2 and 3.4 in K&C, in addition to the note on nonlinear equations before solving these exercises.

Task 1

a) Apply Newton's method to the equation $f(x) = 0$ where

i) $f(x) = \cos(x) - 1/2$, with $x_0 = 0.5$.
ii) $f(x) = e^x - x - 1$, with $x_0 = 0.5$.
iii) $f(x) = x\big(1 - \cos(x)\big)$, with $x_0 = 0.5$.

The iterations will converge to a zero $x^\star$ in all three cases. Measure the order of convergence in the three cases. Is the result in accordance with theory? If no, can you explain why?

We say that a zero $x^\star$ of $f(x)$ has multiplicity m if there exists a function $q(x)$ such that

$$f(x) = (x - x^\star)^m q(x), \qquad q(x^\star) \neq 0,$$

which is the case if and only if

$$f(x^\star) = f'(x^\star) = \cdots = f^{(m-1)}(x^\star) = 0, \qquad f^{(m)}(x^\star) \neq 0.$$

b) What is the multiplicity of the solutions of the three equations in **a)**?

c) Assume that $x^\star$ is a zero with multiplicity m of the function $f(x)$. Show that the function

$$\mu(x) = f(x)/f'(x)$$

has a simple zero in $x^\star$, independent of m. Use this to find an iteration scheme that converges quadratically to $x^\star$.

d) Test the new scheme on the functions *ii)* and *iii)* in **a)**.

Task 2

Given

$$G(x) = \begin{pmatrix} \frac{1}{3}\cos(x_1 x_2) + \frac{1}{6} \\ \frac{1}{9}\sqrt{x_1^2 + \sin(x_3) + 1{,}06} - 0{,}1 \\ -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60} \end{pmatrix}$$

Show that the fixed point iterations $x^{(k+1)} = G(x^{(k)})$ converge towards a unique fixed point for all starting values $x^{(0)}$ in $D = \{x \in \mathbb{R}^3 : -1 \leq x_i \leq 1,\ i = 1, 2, 3\}$.

Verify the result numerically.

Task 3

In the lecture, we discussed the system of equations

$$x_1^2 + x_2^2 = 1,$$
$$x_1^3 - x_2 = 0.$$

This has two solutions, one in the region $-1 \leq x_1, x_2 \leq 0$ and one in $0 \leq x_1, x_2 \leq 1$. We showed numerically that the iteration scheme based on the formulation

$$x_1 = \sqrt[3]{x_2},$$
$$x_2 = \sqrt{1 - x_1^2}$$

converged with appropriate starting values.

Explain why. How would you select starting values?

Hint: It is simpler to analyse results if you consider two subsequent iterations as one.