TMA4215 Numerical Mathematics

Autumn 2009

Exercise 8

Task 1

Consider the linear system of equations:

$$\begin{aligned} x_1 - 5x_2 + x_3 &= 7 \\ 10x_1 + 20x_3 &= 6 \\ 5x_1 - x_3 &= 4 \end{aligned}$$

Solve the equations by

a) Naive Gaussian elimination.

b) Gaussian elimination with scaled row pivoting.

Give the LU factorisation of the matrix in both cases.

Task 2

Given the iteration scheme:

$$\begin{aligned} 4x_{k+1} &= -x_k - y_k + z_k + 2 \\ 6y_{k+1} &= 2x_k + y_k - z_k - 1 \\ -4z_{k+1} &= -x_k + y_k - z_k + 4 \end{aligned}$$

Prove that $\mathbf{x}^{(k)} = [x_k, y_k, z_k]^T$ converges to a limit $\mathbf{x}$ for all starting values $\mathbf{x}^{(0)}$ when $k \to \infty$. What is the limit $\mathbf{x}$?

How many iterations are needed to ensure $\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq 10^{-4}$ if $\mathbf{x}^{(0)} = [0, 0, 0]^T$? Is this number realistic, or do you think that less iterations are needed in practice?

Task 3

Solve the two systems of equations by Gauss–Seidel iterations:

$$\begin{aligned} 3x + y + z &= 5 \\ x + 3y - z &= 3 \\ 3x + y - 5z &= -1 \end{aligned} \tag{1}$$

$$\begin{aligned} 3x + y + z &= 5 \\ 3x + y - 5z &= -1 \\ x + 3y - z &= 3. \end{aligned} \tag{2}$$

Use $[0.1, 0.1, 0.1]^T$ as the starting point. First do a few iterations by hand, and then you can use the enclosed MATLAB-program `gs`.

Comment the results. Do they comply with theory?

Task 4

Given the system of equations:

$$\begin{aligned}
4x_1 - x_2 - x_4 &= 0 \\
-x_1 + 4x_2 - x_3 - x_5 &= 5 \\
-x_2 + 4x_3 - x_6 &= 0 \\
-x_1 + 4x_4 - x_5 &= 6 \\
-x_2 - x_4 + 4x_5 - x_6 &= -2 \\
-x_3 - x_5 + 4x_6 &= 6
\end{aligned}$$

These equations are to be solved by an SOR method. The tasks are:

a) Find an optimal ω and the corresponding $\rho(T_\omega)$. If you prefer, use the MATLAB function `rhoSOR`, and plot $\rho(T_\omega)$ as a function of ω.

b) Do 10 iterations using the optimal ω. For each iteration, print the error $\|\mathbf{x}^{(k)} - \mathbf{x}\|_2$. Hint: Rewrite the routine `gs` to do SOR iterations.

c) Repeat **b)** using other values of ω, e.g. 1.0 and 1.3. How does this affect the rate of convergence compared to the results of **b)**? Is this as expected? Find a value of ω giving $\rho(T_\omega) = 1$, and do iterations using values of ω around this value. How do the results comply with theory?

Task 5

Which of the following linear multistep methods is/are convergent?

Give the order p and the error constant C_{p+1} for each method.

a)

$$y_{m+2} + y_{m+1} - 2y_m = \frac{h}{4}\left[f(x_{m+2}, y_{m+2}) + 8f(x_{m+1}, y_{m+1}) + 3f(x_m, y_m)\right].$$

b)

$$y_{m+3} + \frac{1}{4}y_{m+2} - \frac{1}{2}y_{m+1} - \frac{3}{4}y_m = \frac{h}{8}\left[19f(x_{m+2}, y_{m+2}) + 5f(x_m, y_m)\right].$$

c)

$$y_{m+2} - y_{m+1} = \frac{h}{3}\left[3f(x_{m+1}, y_{m+1}) - 2f(x_m, y_m)\right].$$