TMA4215 Numerical Mathematics

Autumn 2009

Project 1

The project is about developing, testing and implementing an algorithm for solving two-point boundary value problems of the form

$$u'' + p(x)u' + q(x)u = r(x), \qquad x \in [a,b], \qquad u(a) = u_a, \quad u(b) = u_b \tag{1}$$

numerically.

The numerical algorithm is as follows:

1. Choose $n+1$ distinct nodes in $[a,b]$ with $x_0 = a$ and $x_n = b$.
2. Find a polynomial $v \in \mathbb{P}_n$ satisfying the boundary conditions in addition to the differential equation in all inner nodes. This means that v must satisfy

$$v(a) = u_a, \qquad v(b) = u_b \tag{2a}$$

$$v''(x_i) + p(x_i)v'(x_i) + q(x_i)v(x_i) = r(x_i), \qquad i = 1, \dots, n-1. \tag{2b}$$

Thus, we can use $v(x)$ as a numerical approximation to the solution $u(x)$ of (1). The polynomial $v(x)$ can be written on Lagrange form, i.e.

$$v(x) = \sum_{k=0}^{n} v_k \ell_k(x), \qquad \ell_k(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

This means that $v(x)$ is determined by v_i, $i = 0, \dots, n$, which is given by the conditions (2).

Task 0

This Task is meant as an introduction to the problem. You should do it, but it is not *to be handed in.*

Given the equation

$$u'' + u = 2\cos(x), \qquad u(0) = 0,\ u(\pi/2) = \pi/2,$$

with exact solution $u = x\sin(x)$.

a) Choose nodes $x_0 = 0$, $x_1 = \pi/4$ and $x_2 = \pi/2$, and find the polynomial $v(x)$ with the recipe above. Compare v and the exact solution u (make a plot).

b) Add the nodes $x_3 = \pi/8$ and $x_4 = 3\pi/8$, and repeat the experiment.

Task 1

The condition (2b) demands that it is possible to calculate the derivative of the polynomial $v(x)$ in the nodes. We can write this more generally as:

- Let $p \in \mathbb{P}_n$ be given by $p(x_i) = v_i$, $i = 0, 1, \ldots, n$.
- Let $w_i = p'(x_i)$, $i = 0, 1, \ldots, n$.

This is a linear operation that can be written as

$$w = D_n v$$

where $D_n \in \mathbb{R}^{(n+1)\times(n+1)}$, $v = [v_0, \ldots, v_n]^T$, and $w = [w_0, \ldots, w_n]^T$.

a) Find D_2 from the nodes $x_0 = 0,\ x_1 = \pi/4$ and $x_2 = \pi/2$.

b) Given an arbitrary set of discrete nodes $(x_i)_{i=0}^n$. Show that D_n is given by

$$D_{ik} = \begin{cases} \dfrac{a_i}{a_k(x_i - x_k)} & \text{when } i \neq k \\ \displaystyle\sum_{\substack{j=0 \\ j\neq k}}^{n} \dfrac{1}{x_k - x_j} & \text{for } i = k, \end{cases}$$

where $a_k = \displaystyle\prod_{\substack{j=0 \\ j\neq k}}^{n} (x_j - x_k)$.

c) Write a MATLAB function that calculates the differentiation matrix D_n for a given set of nodes x.

d) Let p interpolate a given function $f(x)$ in the nodes $(x_i)_{i=0}^n$. In this case, we can use $w_i = p'(x_i) \approx f'(x_i)$ as an approximation to the derivative in the node. Test this on the functions

$$\begin{aligned} &i) && f(x) = e^x \sin(5x) \\ &ii) && f(x) = 1/(1 + x^2) \\ &iii) && f(x) = |x^3| \\ &iv) && f(x) = x^{10} \end{aligned}$$

on the interval $x \in [-1, 1]$ with nodes given by

$$x_i = -\cos(i\pi/n), \qquad i = 0, \ldots, n. \tag{3}$$

Find the error $E_n = \max_{i=0,\ldots,n} |f'(x_i) - w_i|$ numerically and plot this as a function of n, $n = 1, \ldots, 50$. Comment the result.

e) Let $\bar{w}_i = p''(x_i)$, $i = 0, \ldots, n$. Convince yourself that these coefficients are given by

$$\bar{w} = D_n^2 v.$$

```
% Define the equation:
p = ...
q = ...
r = ...

% Boundary conditions
a = ...
b = ...
ua = ...
ub = ...

% Number of nodes - 1
n = ...

  :
  :

% Plot the solution.
```

Figure 1: Example of the MATLAB code in Task 2.

Task 2

a) Show that the conditions (2) make up a system of linear equations

$$Mv = b.$$

Explain how M can be expressed utilising the differentiation matrix D_n in Task 1.

b) Write a MATLAB script which solves an arbitrary two-point boundary value problem (1) numerically by using (2). Use the node distribution given by (3), perhaps moving them over to the interval $[a, b]$. The script must be of the form shown in Figure 1.

Test the program on the problem in Task 0 first. Choose $n = 10$ and $n = 20$ and plot both the solution and the error.

Then use the program to solve the following equations:

$$\begin{array}{lll} i) & u'' + 10u' + u = 1, & u(-1) = u(1) = 0. \\ ii) & u'' - \dfrac{2x}{x^2+1}u' + \dfrac{2}{x^2+1}u = x^2 + 1, & u(0) = 2,\ u(1) = 5/3. \end{array}$$

You are free to choose more equations.