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Contact during exam:
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

10. December 2006
Time: 09:00–13:00, Grades due: 15.08.2006

Permitted aids: Cathegory B, all written aids permitted.
Simple calculator with empty memory allowed.

Problem 1 Calculate

$$\int_{-1}^{1} \frac{e^x \, dx}{\sqrt{1-x^2}}$$

with atleast 4 significant digits (note the given formulas).

Answer: We can use Gauss-Chebyshev quadrature with $f(x) = e^x$, which yields $f^{(k)}(x) = e^x$ for all k and thus that $|f^{(2n)}(\xi)| \leq e$ for all k. Hence we can give a table listing the error for each n.

n	1	2	3	4	5	6	7
error	2.13	0.04	$0.37 \cdot 10^{-3}$	$0.16 \cdot 10^{-5}$	$0.45 \cdot 10^{-8}$	$0.87 \cdot 10^{-11}$	$0.11 \cdot 10^{-13}$

We see that Q_4 is enough. We have $Q_4(f) = 3.9775$.

$$Q_4(f) = \frac{\pi}{4}\left(\exp(-\cos(\frac{\pi}{8})) + \exp(-\cos(\frac{3\pi}{8})) + \exp(\cos(\frac{3\pi}{8})) + \exp(\cos(\frac{\pi}{8}))\right)$$

Problem 2 The Van der Pol oscillator can be modelled by the initial value problem

$$u'' + \alpha(u^2 - 1)\, u' + u = 0, \qquad u(0) = u_0,\ u'(0) = v_0 \tag{1}$$

The method *backward Euler* (reluE) for the problem $y' = f(y)$ is given as

$$y_{m+1} = y_m + h\, f(y_{m+1}) \tag{2}$$

where h is the step size.

a) Show that (2) applied to (1) leads to the following equation to be solved (with respect to u) for $u_1 \approx u(h)$:

$$\alpha h\, u^3 - \alpha h u_0\, u^2 + (1 - \alpha h + h^2)\, u - u_0(1 - h\alpha) - h v_0 = 0 \tag{3}$$

Answer: First we have to state the problem as a system of first order equations, $u' = v$ and $v' = -\alpha(u^2 - 1)v - u$. If we write v for v_1 and u for u_1 we get

$$u = u_0 + hv, \qquad v = v_0 - \alpha\, h\, (u^2 - 1)\, v - h\, u$$

We now insert $v = (u - u_0)/h$ into the second of these equations and end up with the given answer.

b) Now let $\alpha = 5$, $h = 0.1$, $u_0 = 2$, $v_0 = 0$, and find u_1 with atleast 6 significant digits.

Answer: Newton's method with an intial guess $u^{(0)} = u_0 = 2$ is the best way to solve this. We start by inserting $\alpha = 5$, $u_0 = 2$, $v_0 = 0$, and $h = 0.1$ which leads to

$$F(u) = 0.5\, u^3 - u^2 + 0.51\, u - 1 = 0 \qquad \Rightarrow \qquad F'(u) = 1.5\, u^2 - 2\, u + 0.51.$$

We calculate $u^{(k+1)} = u^{(k)} - F(u^{(k)})/F'(u^{(k)}$ which gives the table below.

k	$u^{(k)}$	$F(u^{(k)})$	$F'(u^{(k)})$
0	2.000000	0.02	2.51
1	1.992032	$0.126729 \cdot 10^{-3}$	2.478222
2	1.991980	$0.519868 \cdot 10^{-8}$	2.478019
3	1.991980	$0.874959 \cdot 10^{-17}$	2.478019

We conclude that $u_2 = 1.991980$ is correct to atleast 6 digits.

c) Find a polynomial $p(x)$ of degree at most 3 which satisfies

$$p(0) = 2,\ p'(0) = 0,\ p(h) = u^*,\ p'(h) = (u^* - 2)/h,$$

where h and u^* are arbitrary parameters. Then use these to approximate $u(0.05)$ for the Van der Pol oscillator with u_0, v_0 and α as in the previous task.

Answer: There are several ways to attack this problem. For instance we can use generalized divided differences. From the left end point we can immediately see that

$$p(\theta h) = 2 + a\theta^3 + b\theta^2 \qquad \Rightarrow \qquad p'(\theta h) = \frac{1}{h}(3a\theta^2 + 2b\theta).$$

Now; $p(h) = 2 + a + b = u^*$ and $p'(h) = \frac{1}{h}(3a + 2b) = \frac{1}{h}(u^* - 2)$. Hence we have that $a = 2 - u^*$ and $b = 2(u^* - 2)$, that is

$$p(\theta h) = 2 + (2 - u^*)\theta^3 + 2(u^* - 2)\theta^2.$$

We can approximate $u(0.05)$ from earlier by using $\theta = \frac{1}{2}$, $h = 0.1$, and $u^* = u_1$ $p(0.05) = 2 + (2 - u_1)/8 + 2(u_1 - 2)/4 = 1.25 + 0.375u_1 = 1.996992$.

Problem 3 The tip of a robotic arm moves along a path in the xy-plane which can be described by a parabola-like graph$(x, f(x))$, that is $f(x)$ can be approximated well by a 2. degree polynomial $p(x)$. The following have been observed.

x_m	-1.0	-0.6	-0.2	0.2	0.6	1.0
y_m	-8.7065e-03	3.2370e-01	9.4428e-01	1.2189	1.1431	1.0584

You can assume that the x-values are exact, while there is some uncertainty in the y-values.

a) Find $p(x) \in \Pi_2$ which minimizes the sum of squares

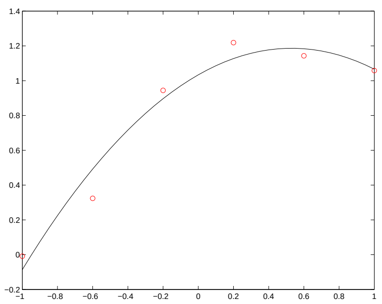
$$E[p] = \sum_{m=1}^{6} (y_m - p(x_m))^2$$

Answer:

We state the normal equations by letting $\phi_k(x) = x^{k-1}, k = 1, 2, 3,$ and defining the 6×3-matrix Φ by $\Phi_{m,k} = \phi_k(x_m)$. We now consider $\Phi^T \Phi\, c = \Phi^T y$

$$\begin{bmatrix} 6 & 0 & 2.8 \\ 0 & 2.8 & 0 \\ 2.8 & 0 & 2.2624 \end{bmatrix} \cdot c = \begin{bmatrix} 4.6797 \\ 1.6137 \\ 1.6643 \end{bmatrix} \quad \Rightarrow \quad c = \begin{bmatrix} 1.0336 \\ 0.5763 \\ -0.5436 \end{bmatrix}$$

Hence $p(x) = 1.0336 + 0.5763\, x - 0.5436\, x^2$.



b) We suspect that two of the data points are rather inexact, namely the second $(-0.6,$ 3.2370e-01$)$, and the fourth $(0.2, 1.2189)$. We want to give these a somewhat reduced weight in the sum of square, which leads us to defining

$$E_w[p] = \sum_{m=1}^{6} w_m (y_m - p(x_m))^2.$$

We now let $w_1 = w_3 = w_5 = w_6 = 1$ and $w_2 = w_4 = \frac{1}{2}$. We can define an inner product on $\mathbf{R}^6$ by

$$\langle u, v \rangle_w = \sum_{m=1}^{6} w_m u_m v_m$$

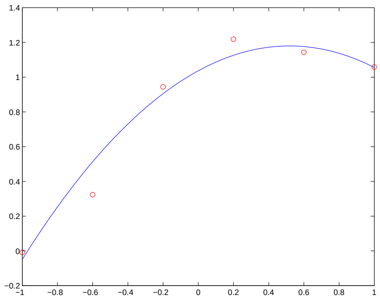
To a polynomial $\phi \in \Pi_2$ we associate the vector $\bar{\phi} = [\phi(x_1), \ldots, \phi(x_6)]^T \in \mathbf{R}^6$. It turns out that the three polynomials

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = x - \frac{1}{25}, \quad \phi_2(x) = x^2 - \frac{13}{25}$$

satisfies $\langle \bar{\phi}_k, \bar{\phi}_\ell \rangle_w = 0$ when $k \neq \ell$. Use this (without proving it) to find $q \in \Pi_2$ which minimizes $E_w[q]$.

Answer: >From the orthogonality principle we get that the sought polynomial $p(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) \in \Pi_2$ satisfies the normal equations

$$\sum_{\ell=0}^{2} c_\ell \langle \bar{\phi}_\ell, \bar{\phi}_k \rangle_w = \langle y, \bar{\phi}_k \rangle_w, \quad k = 0, 1, 2.$$



The fact that $\langle \bar{\phi}_k, \bar{\phi}_\ell \rangle_w = 0$ when $k \neq \ell$, then leads to the rather simple solution

$$c_\ell = \frac{\langle y, \bar{\phi}_\ell \rangle_w}{\langle \bar{\phi}_\ell, \bar{\phi}_\ell \rangle_w},$$

that is $\quad c_0 = 0.7818, \quad c_1 = 0.5527, c_2 = -0.5335$

If we now state $p(x)$ on canonical form, we have

$$p(x) = -0.5335\, x^2 + 0.5527\, x + 1.0370$$

Some useful formulas

1. **Chebyshev-quadrature**

$$I(f) = \int_{-1}^{1} \frac{f(x)\, dx}{\sqrt{1 - x^2}} \approx Q_n(f) = \sum_{k=1}^{n} c_{n,k} f(x_{n,k})$$

$$\begin{aligned} c_{n,k} &= \frac{\pi}{n} \\ x_{n,k} &= \cos \tfrac{2k-2n-1}{2n}\pi \end{aligned}, \quad k = 1, \ldots, n.$$

2. **Error in Chebyshev-quadrature**

$$I(f) - Q_n(f) = \frac{\pi}{(2n)!\, 2^{2n-1}} f^{(2n)}(\xi), \quad -1 < \xi < 1$$