Norwegian University of Science and Technology
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Contact during exam:
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

08. August 2006
Time: 09:00–13:00, Grades due: 15.08.2006

Permitted aids: Cathegory B, all written aids permitted.
Simple calculator with empty memory allowed.

Problem 1 Let $\mathcal{S}$ be the vector space of quadratic splines on the interval $[-1, 1]$ with knots in $-1, 0, 1$.

a) Find a $S \in \mathcal{S}$ that satisfies

$$S(-1) = 0, \quad S(0) = 1, \quad S'(0) = 2, \quad S(1) = 2.$$

Svar: $S(x) = x^2 + 2x + 1$ in $[-1, 0]$ and $S(x) = -x^2 + 2x + 1$ in $[0, 1]$

b) Now, find a $S \in \mathcal{S}$ which satisfies $S(-1) = 0$, $S(0) = 1$, $S(1) = 2$ and such that $\int_{-1}^{1} S(x)^2 \, \mathrm{d}x$ is as small as possible.

Svar: $S(x) = 1 - \frac{3}{2}x - \frac{5}{2}x^2$ for $x \in [-1, 0]$ and $S(x) = 1 - \frac{3}{2}x + \frac{5}{2}x^2$ for $x \in [0, 1]$

Problem 2 The function $f(x)$ is sampled equidistantly in the points $x_k = 1 - 0.1\,k$, the results are tabulated below.

k	0	1	2	3	4	5	6
$f(1 - 0.1\,k)$	0.0000	0.05263	0.1111	0.1000	0.1764	0.2500	0.3333

a) Give the table of backward differences for this data set.

Svar:

0.0000						
	-0.05263					
0.05263		0.005848				
	-0.05848		-0.001032			
0.1111		0.006879		$0.2580 \cdot 10^{-3}$		
	-0.06536		-0.001280		$-.8600 \cdot 10^{-4}$	
0.1000		0.008170		$0.3440 \cdot 10^{-3}$		$.3686 \cdot 10^{-4}$
	-0.07353		-0.001634		$-.1229 \cdot 10^{-3}$	
0.1764		0.009804		$.4669 \cdot 10^{-3}$		
	-0.08333		-0.002101			
0.2500		0.01190				
	-0.09524					
0.3333						

b) Find the interpolation polynomial $p(x)$ of degree 3 based on abscissas as close to $x = 1$ as possible. If you want you you can state the polynomial using backward differences.

Svar: Let $h = 0.1$ and $x = x_n + sh = 1 + 0.1 \cdot s$

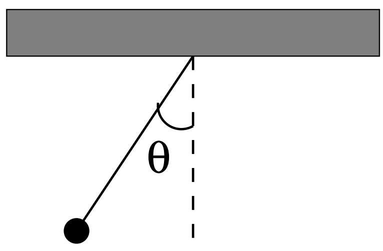
$$p(x) = p(1 + 0.1 \cdot s) = -\binom{-s}{1}$$

c) Assume that the function $f(x)$ have at least 4 continous derivatives. Give an estimate for the error $f(1) - p(1)$.

Problem 3

We want to calculate the angle of deflection $\theta(t)$ in a mathematical pendelum of length ℓ, where t denotes time. In the beginning of the calculation, at $t = 0$, the pendelum has an angle of deflection given by α (where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$) and velocity zero. The function $\theta = \theta(t)$ can then be given implicitly by the equation

$$t = \sqrt{\frac{\ell}{2g}} \int_{\alpha}^{\theta} \frac{\mathrm{d}u}{\sqrt{\cos u - \cos\alpha}} \tag{1}$$



where $g = 9.81\,\mathrm{m/s^2}$ is the gravity constant. Using a change of variable, we can state (1) as

$$t = \sqrt{\frac{\ell}{g}} \int_{-\pi/2}^{\psi} \frac{\mathrm{d}v}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 v}} \tag{2}$$

where

$$\sin \frac{\theta}{2} = -\sin \frac{\alpha}{2} \sin \psi \tag{3}$$

We can now calculate pairs of t and θ values by first using numerical integration on the right hand side of (1) or (2) and then utilizing (3).

a) Explain why it is much better to use the composite trapezoidal rule on (2) rather than (1).

b) Now assume that we use the composite trapezoidal rule on (2) using step size h, and that $-\frac{\pi}{2} < \psi \leq -\frac{\pi}{4}$. Show that we have the following upper bound for the absolute value of the cancellation error in t:

$$|\Delta t| \leq \sqrt{\frac{\ell}{g}} \frac{\pi}{48} \frac{\tan^2 \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \left(1 + \frac{3}{4} \tan^2 \frac{\alpha}{2}\right) h^2.$$

c) Now let $\alpha = -\frac{\pi}{6}$ and $\ell = 1$ meter. Use the trapezoidal rule with step size $h = \frac{\pi}{8}$ in (2) to find t for $\psi = -\frac{3\pi}{8}$ and $\psi = -\frac{\pi}{4}$ respectively. How many significant digits do we have in the answers?