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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

Thursdag 14. Desember 2006
Time: 15:00–19:00, Grades due: 04.01.2007

Permitted aids: Cathegory B, all written aids permitted.
Simple calculator with empty memory allowed.

Problem 1 You are given the function

$$f(x) = \frac{\cos x}{\cosh x} \qquad (\text{where } \cosh x = \frac{e^x + e^{-x}}{2})$$

In the rest of this task you can assume that for all $m \in \mathbb{N}$ we have

$$\max_{0 \le x \le 1} |f^{(m)}(x)| \le M_m \quad \text{der } M_m = m! \cdot 3 \cdot 1.55^{-m}$$

Every positive integer n define $n+1$ abscissa (Chebyshev-points relative to $[0,1]$)

$$x_{n,k} = \sin^2 \frac{(2k+1)\pi}{4(n+1)}, \quad k = 0, \dots, n$$

a) Find the polynomial of degree 2 which interpolates $f(x)$ in the abscissas $x_{2,0}, x_{2,1}, x_{2,2}$. State your answer on the form $p_2(x) = a_0 + a_1 x + a_2 x^2$.

Answer: We first find the three absciassas

$$\frac{1}{2} - \frac{1}{4}\sqrt{3} \approx 0.066987, \ \frac{1}{2} = 0.5, \ \frac{1}{2} + \frac{1}{4}\sqrt{3} \approx 0.93301.$$

You can use the algorithm of your preference, they all lead to the answer

$$p_2(x) := 1.015260568 - 0.266903474\, x - 0.4142081736\, x^2$$

b) Now assume that we want to make sure that the maximal interpolation error in the interval $[0,1]$ is at most 10^{-6} using the abscissas we found above. Find the smallest possible value of n which makes this possible.

Answer: Here it is important to note that the abscissas utilitzed are Chebyshev points relative to $[0,1]$, that is the given $x_{n,k}$ are the roots of the polynomial $T_{n+1}(2\,x-1)$. We use the bound

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!)} \prod_{k=0}^{n} (x - x_{n,k})$$

Thus the product on the right hand side needs to be a constant multiplied with the polynomial $T_{n+1}(2x-1)$. We recall that $T_{n+1}(y) = 2^n y^{n+1} + \cdots$, hence

$$T_{n+1}(2x-1) \;=\; 2^n(2x-1)^{n+1} + \cdots \;=\; 2^{2n+1}x^{n+1} + \cdots$$

so the leading coefficient is 2^{2n+1}. This means that the product given above is given by $2^{-2n-1}\,T_{n+1}(2x-1)$ and the maximal value of this on the interval $[0,1]$ is 2^{-2n-1}.

We now use the given bound for $|f^{(m)}(x)|$ on $[0,1]$ with $m = n+1$ which yields

$$\max_{0\le x\le 1} |f(x) - p_n(x)| \le 3 \cdot 1.55^{-n-1} \cdot 2^{-2n-1}$$

To have a maximal error of at most ε we need

$$3 \cdot 1.55^{-n-1} \cdot 2^{-2n-1} \le \varepsilon.$$

If we now take the logarithm (a monotone increasing function for positive arguments) on both sides and solve with respect to n;

$$n \ge 0.55 \cdot \ln\left(\frac{0.97}{\varepsilon}\right).$$

Since this is not an integer in general, we have to round up to the closest integer. With $\varepsilon = 10^{-6}$ we find $n = 8$ after rounding. Note! If you had used "unscaled" Chebyshev polynomials you would end up with $n = 13$.

c) Calculate the value of the integral

$$\int_0^1 \frac{\cos x}{\cosh x}\,\mathrm{d}x$$

with an error which is guaranteed to be less than 10^{-4}. Justify your answer and your solution strategy.

Answer: Here you are free to use any method you like. However note the way the task is formulated, we cannot simply use an error estimate, we need to use an error bound. This is not very involved here since we have been given bounds for the m'th derivative. We here use Simpson's rule. Basically we have to decide how many subintervals we need. The error bound for (composite) Simpson's rule is given in the book,

$$|I(f) - S_n(f)| \le \frac{1}{180}\, h^4 \max_{0\le x\le 1} |f^{(4)}(x)|.$$

Inserting the given bounds leads to $0.07h^4 \leq 10^{-4}$ which yields approximately $h \approx 0.195$. Hence we need to use atleast $n = 6$ subintervals (recall that n needs to be even). Using $h = 1/6$ we get

$$S_6(f) = \frac{h}{3}(f(0) + 4f(\frac{1}{6}) + 2f(\frac{1}{3}) + 4f(\frac{1}{2}) + 2f(\frac{2}{3}) + 4f(\frac{5}{6}) + f(1)) \approx 0.7437$$

The equivalent bounds for the trapezoidal rule and the midpoint rule would have given 46 and 34 subintervals respectively, which probably is a bit too much work on an exam.

Problem 2 In this exercise we study the formula

$$N_h(f)(x) = \frac{1}{h^3}\left(-\frac{1}{2}f(x-2h) + f(x-h) - f(x+h) + \frac{1}{2}f(x+2h)\right) \tag{1}$$

used to approximate $f^{(3)}(x)$ (the third derivative of f) for a smooth function f.

a) Test the formula on $f(x) = \sin x$ with $x = 0$ and $h = 0.1$.

Answer: We can use the fact that $\sin(x)$ is odd

$$N_h(\sin)(x) = 0.1^{-3}(\sin(0.2) - 2\sin 0.1) \approx -0.9975$$

Note that the exact answer is $-\cos 0 = -1$.

b) Show that

$$N_h(f)(x) = f^{(3)}(x) + K_2h^2 + K_4h^4 + \cdots,$$

that is, only even powers of h. Also find an expression for K_2.

Answer: One can argue for the even expansion by pointing out, then utilizing, the fact that $N_{-h}(f) = N_h(f)$. We have

$$N_h(f) = \sum_{j=0}^{\infty} K_j\, h^j \quad \Rightarrow N_{-h}(f) = \sum_{j=0\infty} (-1)^j K_j\, h^j$$

such that

$$N_h(f) = \frac{1}{2}(N_h(f) + N_{-h}(f)) = \sum_{m=0}^{\infty} K_{2m}\, h^{2m}.$$

In particular we can consider the h^0- and h^2-contribution by including the h^3- and h^5-terms respectively in the Tayler series around $h = 0$ for each $f(x \pm h), f(x \pm 2h)$. We get

$$\frac{1}{2}(f(x+2h) - f(x-2h)) = 2hf'(x) + \frac{4}{3}h^3f^{(3)}(x) + \frac{4}{15}h^5f^{(5)}(x) + \cdots$$

and

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{1}{3}h^3f^{(3)}(x) + \frac{1}{60}h^5f^{(5)}(x) + \cdots$$

If we take the difference between these two and divide by h^3, we find that the coefficient in front of the h^0 term is $K_0 = f^{(3)}(x)$, while

$$K_2 = \frac{1}{4}\, f^{(5)}(x).$$

c) Somebody has utilized formula (1) on a smooth function $f(x)$ which yielded the following table for a given x

h	0.32	0.16	0.08
$N_h(f)(x)$	-0.250118	-0.253512	-0.253933

Use this information to make a better (optimal) approximation to $f^{(3)}(x)$.

Answer: The keyword here is Richardson extrapolation. Since we have an even expansion of the error, we can use a recursion formula just like in Romberg intergration, say

$$Q_{k,j} = Q_{k,j-1} + \frac{Q_{k,j-1} - Q_{k-1,j-1}}{4^{j-1} - 1}.$$

where we put $Q_{k,1} = N_{h_k}(f)$ with $h_k = \frac{h_1}{2^{k-1}}$. In our case we have $h_1 = 0.32$. The table reads

-0.250118		
-0.253512	-0.254643	
-0.253933	-0.254073	-0.254035

i.e the answer is -0.254035.

Problem 3 We here consider the linear system of two ordinary differential equations given by

$$y' = A\,y, \qquad \text{where} \quad A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}, \qquad y(0) = y_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{2}$$

a) Let $h = 0.4$, $u_0 = v_0 = 1$ and apply one step with Euler's method to this problem.

Answer: We get $u_1 = 0.6$ and $v_1 = -0.2$.

We now, as usual, let the 2-norm of a vector $y = [u, v]^T$ be defined by

$$\|y\|_2 = \sqrt{u^2 + v^2}.$$

It is easy to show that for the exact solution of (2) we have

$$\|y(x)\|_2 = \mathrm{e}^{-2x}\,\|y_0\|_2, \quad \text{for all } y_0.$$

b) Show that if y_1, y_2, $y_3, \ldots$ are the approximations obtained by applying Euler's method with step size h to (2), we have

$$\|y_{n+1}\|_2 = \sqrt{5h^2 - 4h + 1}\,\|y_n\|_2.$$

Find the largest possible step size H such that for all $0 < h < H$ we have $y_n \to \vec{0}$ when $n \to \infty$.

Answer: It is smart to write $A = -2I + S$ where I is the identify matrix and S is the skew symmetric matrix

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{such that} \quad S^T S = I \quad \text{and } x^T S x = 0 \text{ for all } x \in \mathbf{R}^2.$$

Using the properties of the matrix S, we can now calculate (with $y_{n+1} = (I + hA)y_n$ from Euler's method)

$$\|y_{n+1}\|_2^2 = y_{n+1}^T y_{n+1} = y_n^T((1-2h)I + hS)^T((1-2h)I + hS)\, y_n = ((1-2h)^2 + h^2) y_n^T y_n.$$

The given equality now follows by taking the square root on each side. We now seek the largest possible interval $E = (0, H)$ such that $f(h) = 5h^2 - 4h + 1 < 1$ for $h \in E$.

$$5h^2 - 4h + 1 < 1 \quad \Rightarrow \quad h(5h-4) < 0 \quad \Rightarrow \quad 0 < h < H = 0.8$$

For $h \in (0, 0.8)$ we have $\|y_{n+1}\|_2 = \alpha(h)\|y_n\|_2$ with $\alpha(h) < 1$. Hence $\|y_n\|_2 = \alpha(h)^n \|y_0\|_2 \to 0$ when $n \to \infty$. Note that for all $h > H$ the numerical solution grows to infinity for increasing n (obviously disregarding the null solution).