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Contact during exam:
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

Tuesday December 4 2007
Time: 15:00 – 19:00 Final grades: January 4.

Permitted aids (code B):
All printed and hand written aids.
Approved calculator.

Problem 1 The expected life time t of an industrial fan at different temperatures T is given by

Temperature (℃)	30	40	50	60
Life time($\times$1000 hours)	91	75	63	54

Find the third order polynomial $p(T)$ which interpolates this data set. Use the polynomial to approximate the expected life time at 55℃.

Answer: You can use the algorithm you prefer, the resulting polynomial is

$$t(T) = -\frac{1}{6000}T^3 + \frac{1}{25}T^2 - \frac{227}{60}T + 173$$

such that

$$t(55) = \frac{931}{16} = 58.2 \quad (\times 1000 \text{ hours}).$$

Problem 2 In this problem we study an implicit multi step method given by

$$y_{m+2} - (1+a)y_{m+1} + ay_m = h\left[f_{m+2} - \frac{1+a}{2}f_{m+1} + \frac{1-a}{2}f_m\right], \tag{1}$$

where $f_l = f(x_l, y_l)$ and a is a real number.

a) Find the order of the method and give the error constant for all values of a.

Answer: Inserting the given coefficients into the order conditions for LMM's we find

$$C_0 = C_1 = C_2 = 0, \qquad C_3 = \frac{a-7}{12}.$$

Thus the method is of second order for $a \neq 7$. For $a = 7$ the method is of order 3, and the error constant is $-1/3$.

b) A student wants to test the method by applying it to find the solution of the equation

$$y' = -y^2, \qquad y(0) = 1. \tag{2}$$

at $t = 1$. She uses $h = 0.1$ and the exact solution $y_0 = 1$ and $y_1 = 1/(1+h)$ as initial values. The nonlinear equation in y_{m+2} is solved to machine precision at each step.

The results for two different values of a are given in the table below. Since this is a test, the absolute value of the errors are also given.

	$a=0$		$a=7$	
x_m	y_m	$\lvert y(x_m) - y_m \rvert$	y_m	$\lvert y(x_m) - y_m \rvert$
0.0	1.0000	0	1.0000	0
0.1	0.9091	0	0.9091	0
0.2	0.8313	$2.0271 \cdot 10^{-3}$	0.8338	$4.5256 \cdot 10^{-4}$
0.3	0.7659	$3.3506 \cdot 10^{-3}$	0.7729	$3.6924 \cdot 10^{-3}$
0.4	0.7102	$4.0709 \cdot 10^{-3}$	0.7397	$2.5408 \cdot 10^{-2}$
0.5	0.6622	$4.4176 \cdot 10^{-3}$	0.8354	$1.6871 \cdot 10^{-1}$
0.6	0.6205	$4.5396 \cdot 10^{-3}$	1.6697	$1.0447 \cdot 10^{+0}$
0.7	0.5837	$4.5266 \cdot 10^{-3}$	5.6463	$5.0581 \cdot 10^{+0}$
0.8	0.5511	$4.4332 \cdot 10^{-3}$	17.2646	$1.6709 \cdot 10^{+1}$
0.9	0.5220	$4.2932 \cdot 10^{-3}$	42.9463	$4.2420 \cdot 10^{+1}$
1.0	0.4959	$4.1278 \cdot 10^{-3}$	97.5861	$9.7086 \cdot 10^{+1}$

Explain the obtained results. Which value of a would you recommend?

Answer: The error after one step is less for $a = 7$ compared to the case with $a = 0$, which is in accordance with the discussion in **a)**. The increasing error for $a = 7$ indicate stability problems. We have shown that the method is consistent, however it turns out that it is not necessarily null-stable. The characteristic polynomial for the method is given by

$$\rho(r) = r^2 - (1+a)r + a = (r-1)(r-a)$$

which means the method is null-stable only for $-1 \le a < 1$. This explains the rubbish answers we get using $a = 7$. In conclusion; you should definitely advice the student to choose values of a such that the method is null-stable. In addition, the error constant is smaller the closer we get to $a = 1$.

c) Construct a predictor-corrector method with (1) as corrector and the "leap frog" method

$$y_{m+2} - y_m = 2hf(x_{m+1}, y_{m+1})$$

as predictor. Use $a = 0$. Apply the obtained method to find an approximation of (2) at $t = 2h$. Use $h = 0.1$ and the exact initial values. Also give an estimate of the local truncation error.

Answer: The predictor-correct method reads

$$\begin{aligned} y_{m+2}^{[0]} &= y_m + 2hf_{m+1} \\ y_{m+2}^{[i]} &= hf(x_{m+2}, y_{m+2}^{[i-1]}) + y_{m+1} - \frac{h}{2}(f_{m+1} - f_m), \qquad i = 1, 2, \cdots . \end{aligned}$$

We obtain an error estimate by using Milne's device (Chapter 3.5 in Owren's lecture notes):

$$h\tau_{m+2} \approx \frac{C_3}{C_3^* - C_3}(y_{m+2} - y_{m+2}^{[0]})$$

where $C_3 = -7/12$ and $C_3^* = 1/3$ (the error constant for the predictor C_3^* is given in the notes, or you can find it yourself). Since the predictor and corrector has the same order, one iteration should be sufficient (of course nothing is wrong with using more than one, except you doing unnecessary work). Inserting the results we get

$$y_2^{[0]} = 0.834710, \qquad y_2^{[1]} = 0.83074, \qquad h\tau_2 \approx 2.52 \cdot 10^{-3}.$$

Problem 3 Let $P_s(x)$ be the monic Legendre polynomial of degree s, and define

$$R_s(x) = P_s(x) + \frac{s}{2s-1}P_{s-1}(x).$$

$R_s(x)$ has s distinct, real roots x_i in the interval $[-1, 1]$. These roots can be used to construct quadrature formulas

$$Q_s(f) = \sum_{i=1}^{s} A_i f(x_i) \approx \int_{-1}^{1} f(x)\,\mathrm{d}x = I(f),$$

such that $Q_s(p) = I(p)$ for all polynomials p of degree less than s.

a) Let $s = 2$, and find $Q_2(f) = A_1 f(x_1) + A_2 f(x_2)$.
What is the degree of precision of $Q_2(f)$?

Answer: For $s = 2$ we have $R_2(x) = (x+1)(x-1/3)$, and the quadrature formula reads

$$Q_2(f) = f(-1)\int_{-1}^{1} \frac{x-1/3}{-1-1/3}dx + f(1/3)\int_{-1}^{1} \frac{x+1}{1/3+1}dx = \frac{1}{2}f(-1) + \frac{3}{2}f(1/3)$$

The method is of order 2 (it integrates 2. degree polynomials exactly).

b) Use Q_2 to approximate the integral $\int_{t_j}^{t_j+h} f(t)\,\mathrm{d}t$.
Proceed by using this to construct a composite quadrature formula based on

$$\int_a^b f(t)\,\mathrm{d}t = \sum_{j=0}^{n-1} \int_{t_j}^{t_j+h} f(t)\,\mathrm{d}t, \qquad t_j = a + jh, \quad h = \frac{b-a}{n}.$$

Give the explicit expression for the error in the composite formula.
(Given: $\int_{-1}^{1} f(x)\,\mathrm{d}x - Q_2(f) = \frac{2}{27} f^{(3)}(\xi), \quad \xi \in (-1,1)$.)

Answer: We use the relation $t = t_j + h(1+x)/2$. This yields

$$\begin{aligned}\int_{t_j}^{t_j+h} f(t)dt &= \frac{h}{2}\int_{-1}^{1} f\left(t_j + h\frac{1+x}{2}\right)dx \\ &= \frac{h}{4}\left(f(t_j) + 3f\left(t_j + \frac{2h}{3}\right)\right) + \frac{h^4}{27\cdot 8}\frac{d^3}{dt^3}f(\eta_j), \qquad \eta_j \in (t_j, t_j+h)\end{aligned}$$

Hence the composite formula reads

$$\int_a^b f(t)dt = \frac{h}{4}\sum_{j=0}^{n-1}\left[f(t_j) + 3f\left(t_j + \frac{2h}{3}\right)\right] + \frac{b-a}{216}h^3\frac{d^3}{dt^3}f(\eta) \qquad \eta \in (a,b).$$

c) Use the quadrature formula from task **b**) with $n = 2$ to approximate the integral

$$\int_0^1 \frac{1}{1+t}\,\mathrm{d}t.$$

Also give an upper bound for the error.
What is the value of n needed to guarantee that the error is less than 10^{-5} ?

(If you did not obtain the answer to task b), use Simpson's composite formula with $n = 4$ instead.)

Answer: We have $n = 2$, that is $h = 0.5$. Hence we find $f^{(3)}(t) = -6/(1+t)^4$. By insertion:

$$\int_0^1 \frac{1}{1+t}dt = 0.694129 - \frac{0.5^3}{216}\frac{6}{(1+\eta)^4}.$$

The error is at most

$$|E(f)| \leq \frac{0.5^3}{216} \max_{\eta \in (0,1)} \frac{6}{(1+\eta)^4} = 3.47 \cdot 10^{-3}.$$

d) Show that $Q_s(f)$ has degree of precision $2s-2$.

Answer: The proof is basically as the proof of Theorem 4.7 in B&F. From the construction of Q_s we know that $Q_s(f)$ has atleast degree of precision $s-1$. We also know that $\int_{-1}^{1} R_s(x)p(x)dx = 0$ for all $p \in \mathbb{P}_{s-2}$. Now, let$p$ be a polynomial of degree higher than or equal to s but less than $2s-2$. Polynomial division yields ($R_s(x) \in \mathbb{P}_s$):

$$p(x) = q(x)R_s(x) + r(x), \qquad q, r \in \mathbb{P}_{s-2}$$

We get

$$\int_{-1}^{1} p(x)dx = \int_{-1}^{1} q(x)R_s(x)dx + \int_{-1}^{1} r(x)dx = \int_{-1}^{1} r(x)dx.$$

The first term disappears since $q \in \mathbb{P}_{s-2}$.

$$Q_s(p) = \sum_{i=1}^{s} A_i \left(q(x_i)R_s(x_i) + r(x_i)\right) = \sum_{i=1}^{s} A_i r(x_i) = \int_{-1}^{1} r(x)dx.$$

Now, we have $R_s(x_i) = 0$ and Q_s has degree of precision atleast $s-1$.