Norges teknisk–naturvitenskapelige universitet
Institutt for matematiske fag

EKSAMEN I NUMERISK MATEMATIKK (TMA4215)

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FASIT

Oppgave 1 The table of divided differences becomes

-1.00	0.25			
		-0.75		
0.00	-0.5		1.5	
		1.5		1.0
0.50	0.25		3.5	
		5.0		
1.00	2.75			

The interpolation polynomial becomes

$$p(x) = 1.0(x-0.5)x(x+1) + 1.5x(x+1) - 0.75(x+1) + 0.25 = x^3 + 2.00x^2 + 0.25x - 0.5.$$

You could have used Lagrange polynomials as well.

Oppgave 2 With $f(x) = \sqrt{x}\sin(x)$ and $H = b - a = 1$we get

$$R_{11} = \frac{1}{2}(f(0) + f(1)) = 0.4207354924$$

$$R_{21} = \frac{1}{2}R_{11} + \frac{1}{2}f(0.5) = 0.3798702709$$

$$R_{31} = \frac{1}{2}R_{21} + \frac{1}{4}(f(0.25) + f(0.75)) = 0.3684397509$$

The Romberg table becomes

0.4207354924		
0.3798702709	0.3662485304	
0.3684397509	0.3646295776	0.3645216474

The Romberg algorithm depends on a sufficient amount of terms in the Euler-Maclaurin formula, that is f has to be sufficiently differentiable on $[a, b]$. We know that $\sin \in C^\infty[0, 1]$ and Romberg behaves as expected in this case. With $f = \sqrt{x}\sin(x)$, we get $f'(x) = (\sin(x) + 2\cos(x))/(2\sqrt{x})$ which is bounded in $[0, 1]$. But $f''(x) = (x\cos(x) - x^2\sin(x) - \sin(x)/4)/(x\sqrt{x})$ which is certainly not bounded when $x = 0$. Thus, we can not expect the Romberg algorithm to work at all.

Oppgave 3

a)

$$\phi_0 = 1, \qquad \langle x\phi_0, \phi_0\rangle = \frac{2}{3}, \qquad \langle \phi_0, \phi_0\rangle = 2, \qquad B_1 = \frac{1}{3},$$
$$\phi_1 = x - \frac{1}{3}, \qquad \langle x\phi_1, \phi_1\rangle = \frac{88}{945}, \qquad \langle \phi_1, \phi_1\rangle = \frac{8}{45}, \qquad B_2 = \frac{11}{21}, \qquad C_2 = \frac{4}{45}$$
$$\phi_2 = (x - \frac{11}{21})(x - \frac{1}{3}) - \frac{4}{45} = x^2 - \frac{6}{7}x + \frac{3}{35}.$$

b) As nodes, we use the zeros of ϕ_2, that is

$$x_1 = \frac{3}{7} - \frac{2}{35}\sqrt{30}, \qquad x_2 = \frac{3}{7} + \frac{2}{35}\sqrt{30}.$$

The weights are given as

$$A_1 = \int_0^1 \frac{1}{\sqrt{x}} \frac{x - x_2}{x_1 - x_2} dx = 1 + \frac{\sqrt{30}}{18}, \qquad A_2 = \int_0^1 \frac{1}{\sqrt{x}} \frac{x - x_1}{x_2 - x_1} dx = 1 - \frac{\sqrt{30}}{18}.$$

With $f(x) = x\sin(x)$ we get

$$A_1 f(x_1) + A_2 f(x_2) = 0.365849.$$

Oppgave 4

a) Let $x_1 = y$ and $x_2 = y'$. The ODE is then

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= 1.0 \\ x_2' &= x_1 x_2 & x_2(0) &= 0.5. \end{aligned}$$

Let $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{x}_0 = [1.0, 0.5]^T$. One step with stepsize $h = 0.1$ is given by

$$\mathbf{k}_1 = \mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \qquad \mathbf{x}_0 + \frac{h}{2}\mathbf{k}_1 = \begin{bmatrix} 1.025 \\ 0.525 \end{bmatrix}$$

$$\mathbf{k}_2 = \mathbf{f}(\mathbf{x}_0 + \frac{h}{2}\mathbf{k}_1) = \begin{bmatrix} 0.525 \\ 0.5381 \end{bmatrix}$$

$$\mathbf{x_1} = \mathbf{x}_0 + h\mathbf{k}_2 = \begin{bmatrix} 1.0525 \\ 0.5538 \end{bmatrix}$$

So $y(0.1) \approx 1.0525$ and $y'(0.1) \approx 0.5538$.

b) The order conditions becomes

$$\begin{aligned} b_1 + b_2 + b_3 &= 1 \\ b_2\frac{1}{2} + b_3 &= \frac{1}{2} \\ b_2\frac{1}{4} + b_3 &= \frac{1}{3} \\ b_3 a_{32}\frac{1}{2} &= \frac{1}{6}. \end{aligned}$$

The solution is $b_1 = b_3 = 1/6$, $b_2 = 2/3$, $a_{32} = 2$, $a_{31} = -1$.

c) Combined with the results from point **a)** we get

$$\mathbf{x}_0 + h(-\mathbf{k}_1 + 2\mathbf{k}_2) = \begin{bmatrix} 1.055 \\ 0.5576 \end{bmatrix}, \qquad \mathbf{k}_3 = \mathbf{f}(\mathbf{x}_0 + h(-\mathbf{k}_1 + 2\mathbf{k}_2)) = \begin{bmatrix} 0.5576 \\ 0.5583 \end{bmatrix}.$$

- The error estimate is given by

$$le_1 = h\sum_{i=1}^{3}(\tilde{b}_i - b_i)\mathbf{k}_i = \begin{bmatrix} 1.27 \cdot 10^{-4} \\ 2.00 \cdot 10^{-4} \end{bmatrix}$$

  so $\|le\|_\infty = 2.00 \cdot 10^{-4}$.
- The step is accepted.
- The stepsize is adjusted by the formula (p=2)

$$h_{new} = P \cdot \sqrt[p+1]{\frac{Tol}{\|le_1\|_\infty}} h = 0.128.$$

Oppgave 5

a) On matrix form, this is

$$\begin{bmatrix} R_1+R_5+R_6 & -R_5 & -R_6 & 0 \\ -R_5 & R_2+R_5+R_7+R_8 & -R_7 & -R_8 \\ -R_6 & -R_7 & R_3+R_6+R_7+R_9 & -R_9 \\ 0 & -R_8 & -R_9 & R_4+R_8+R_9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

The matrix is strictly diagonally dominant.

b)

$$\begin{aligned} 40I_1 - 15I_2 - 15I_3 \qquad\quad &= 6 \\ -15I_1 + 55I_2 - 15I_3 - 15I_4 &= 0 \\ -15I_1 - 15I_2 + 70I_3 - 15I_4 &= 0 \\ -15I_2 - 15I_3 + 55I_4 &= 6 \end{aligned}$$

The Gauss-Seidel iterations results in

$$I^{(0)} = \begin{bmatrix} 0.6 \\ 0.0 \\ 0.0 \\ 0.24 \end{bmatrix}, \qquad I^{(1)} = \begin{bmatrix} 0.15000 \\ 0.10636 \\ 0.10636 \\ 0.16711 \end{bmatrix}, \qquad I^{(2)} = \begin{bmatrix} 0.22977 \\ 0.13725 \\ 0.11446 \\ 0.17774 \end{bmatrix}$$