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Contact during exam::
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

Thursdag 14. Desember 2006
Time: 15:00–19:00, Grades due: 04.01.2007

Permitted aids: Cathegory B, all written aids permitted.
Simple calculator with empty memory allowed.

Problem 1 You are given the function

$$f(x) = \frac{\cos x}{\cosh x} \qquad \text{(where } \cosh x = \frac{e^x + e^{-x}}{2}\text{)}$$

In the rest of this task you can assume that for all $m \in \mathbb{N}$ we have

$$\max_{0 \le x \le 1} |f^{(m)}(x)| \le M_m \quad \text{der } M_m = m! \cdot 3 \cdot 1.55^{-m}$$

Every positive integer n define $n+1$ abscissa (Chebyshev-points relative to $[0,1]$)

$$x_{n,k} = \sin^2 \frac{(2k+1)\pi}{4(n+1)}, \quad k = 0, \dots, n$$

a) Find the polynomial of degree 2 which interpolates $f(x)$ in the abscissas $x_{2,0}, x_{2,1}, x_{2,2}$. State your answer on the form $p_2(x) = a_0 + a_1 x + a_2 x^2$.

b) Now assume that we want to make sure that the maximal interpolation error in the interval $[0,1]$ is at most 10^{-6} using the abscissas we found above. Find the smallest possible value of n which makes this possible.

c) Calculate the value of the integral

$$\int_0^1 \frac{\cos x}{\cosh x}\,\mathrm{d}x$$

with an error which is guaranteed to be less than 10^{-4}. Justify your answer and your solution strategy.

Problem 2 In this exercise we study the formula

$$N_h(f)(x) = \frac{1}{h^3}\left(-\frac{1}{2}f(x-2h) + f(x-h) - f(x+h) + \frac{1}{2}f(x+2h)\right) \tag{1}$$

used to approximate $f^{(3)}(x)$ (the third derivative of f) for a smooth function f.

a) Test the formula on $f(x) = \sin x$ with $x = 0$ and $h = 0.1$.

b) Show that

$$N_h(f)(x) = f^{(3)}(x) + K_2 h^2 + K_4 h^4 + \cdots,$$

that is, only even powers of h. Also find an expression for K_2.

c) Somebody has utilized formula (1) on a smooth function $f(x)$ which yielded the following table for a given x

h	0.32	0.16	0.08
$N_h(f)(x)$	−0.250118	−0.253512	−0.253933

Use this information to make a better (optimal) approximation to $f^{(3)}(x)$.

Problem 3 We here consider the linear system of two ordinary differential equations given by

$$y' = A\,y, \qquad \text{where} \quad A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}, \qquad y(0) = y_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{2}$$

a) Let $h = 0.4$, $u_0 = v_0 = 1$ and apply one step with Euler's method to this problem.

We now, as usual, let the 2-norm of a vector $y = [u, v]^T$ be defined by

$$\|y\|_2 = \sqrt{u^2 + v^2}.$$

It is easy to show that for the exact solution of (2) we have

$$\|y(x)\|_2 = \mathrm{e}^{-2x}\,\|y_0\|_2, \quad \text{for all } y_0.$$

b) Show that if $y_1, y_2, y_3, \ldots$ are the approximations obtained by applying Euler's method with step size h to (2), we have

$$\|y_{n+1}\|_2 = \sqrt{5h^2 - 4h + 1}\ \|y_n\|_2.$$

Find the largest possible step size H such that for all $0 < h < H$ we have $y_n \to \vec{0}$ when $n \to \infty$.