Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam:
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

Tuesday December 4 2007
Time: 15:00 – 19:00 Final grades: January 4.

Permitted aids (code B):
All printed and hand written aids.
Approved calculator.

Problem 1 The expected life time t of an industrial fan at different temperatures T is given by

Temperature (℃)	30	40	50	60
Life time($\times$1000 hours)	91	75	63	54

Find the third order polynomial $p(T)$ which interpolates this data set. Use the polynomial to approximate the expected life time at 55℃.

Problem 2 In this problem we study an implicit multi step method given by

$$y_{m+2} - (1+a)y_{m+1} + ay_m = h\left[f_{m+2} - \frac{1+a}{2}f_{m+1} + \frac{1-a}{2}f_m\right], \tag{1}$$

where $f_l = f(x_l, y_l)$ and a is a real number.

a) Find the order of the method and give the error constant for all values of a.

b) A student wants to test the method by applying it to find the solution of the equation

$$y' = -y^2, \qquad y(0) = 1. \tag{2}$$

at $t = 1$. She uses $h = 0.1$ and the exact solution $y_0 = 1$ and $y_1 = 1/(1+h)$ as initial values. The nonlinear equation in y_{m+2} is solved to machine precision at each step.

The results for two different values of a are given in the table below. Since this is a test, the absolute value of the errors are also given.

	$a=0$		$a=7$	
x_m	y_m	$\lvert y(x_m) - y_m\rvert$	y_m	$\lvert y(x_m) - y_m\rvert$
0.0	1.0000	0	1.0000	0
0.1	0.9091	0	0.9091	0
0.2	0.8313	$2.0271 \cdot 10^{-3}$	0.8338	$4.5256 \cdot 10^{-4}$
0.3	0.7659	$3.3506 \cdot 10^{-3}$	0.7729	$3.6924 \cdot 10^{-3}$
0.4	0.7102	$4.0709 \cdot 10^{-3}$	0.7397	$2.5408 \cdot 10^{-2}$
0.5	0.6622	$4.4176 \cdot 10^{-3}$	0.8354	$1.6871 \cdot 10^{-1}$
0.6	0.6205	$4.5396 \cdot 10^{-3}$	1.6697	$1.0447 \cdot 10^{+0}$
0.7	0.5837	$4.5266 \cdot 10^{-3}$	5.6463	$5.0581 \cdot 10^{+0}$
0.8	0.5511	$4.4332 \cdot 10^{-3}$	17.2646	$1.6709 \cdot 10^{+1}$
0.9	0.5220	$4.2932 \cdot 10^{-3}$	42.9463	$4.2420 \cdot 10^{+1}$
1.0	0.4959	$4.1278 \cdot 10^{-3}$	97.5861	$9.7086 \cdot 10^{+1}$

Explain the obtained results. Which value of a would you recommend?

c) Construct a predictor-corrector method with (1) as corrector and the "leap frog" method

$$y_{m+2} - y_m = 2hf(x_{m+1}, y_{m+1})$$

as predictor. Use $a = 0$. Apply the obtained method to find an approximation of (2) at $t = 2h$. Use $h = 0.1$ and the exact initial values. Also give an estimate of the local truncation error.

Problem 3 Let $P_s(x)$ be the monic Legendre polynomial of degree s, and define

$$R_s(x) = P_s(x) + \frac{s}{2s-1} P_{s-1}(x).$$

$R_s(x)$ has s distinct, real roots x_i in the interval $[-1, 1]$. These roots can be used to construct quadrature formulas

$$Q_s(f) = \sum_{i=1}^{s} A_i f(x_i) \approx \int_{-1}^{1} f(x)\,\mathrm{d}x = I(f),$$

such that $Q_s(p) = I(p)$ for all polynomials p of degree less than s.

a) Let $s = 2$, and find $Q_2(f) = A_1 f(x_1) + A_2 f(x_2)$.
What is the degree of precision of $Q_2(f)$?

b) Use Q_2 to approximate the integral $\int_{t_j}^{t_j+h} f(t)\,\mathrm{d}t$.
Proceed by using this to construct a composite quadrature formula based on

$$\int_a^b f(t)\,\mathrm{d}t = \sum_{j=0}^{n-1} \int_{t_j}^{t_j+h} f(t)\,\mathrm{d}t, \qquad t_j = a + jh, \quad h = \frac{b-a}{n}.$$

Give the explicit expression for the error in the composite formula.
(Given: $\int_{-1}^{1} f(x)\,\mathrm{d}x - Q_2(f) = \frac{2}{27} f^{(3)}(\xi), \quad \xi \in (-1, 1)$.)

c) Use the quadrature formula from task **b**) with $n = 2$ to approximate the integral

$$\int_0^1 \frac{1}{1+t}\,\mathrm{d}t.$$

Also give an upper bound for the error.
What is the value of n needed to guarantee that the error is less than 10^{-5} ?

(If you did not obtain the answer to task b), use Simpson's composite formula with $n = 4$ instead.)

d) Show that $Q_s(f)$ has degree of precision $2s - 2$.