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Contact during exam:
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EXAM IN NUMERICAL MATHEMATICS (TMA4215)

Saturday December 19, 2009
Time: 09:00 – 13:00 Final grades: January 15..

Permitted aids (kode B):
All printed and hand written aids.
Approved calculator.

Give sufficient arguments and intermediate calculations to make it clear how the problems are solved.

Problem 1 Find an approximation to the integral

$$\int_1^2 x^2 \ln x \, dx$$

by Simpson's composite formula. Use $h = 0.25$.

Give an upper limit for the error.

Problem 2 Given the linear equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

in which

$$A = \begin{bmatrix} 1.0 & 1.0 & 0.1 & 0.1 \\ 0.1 & 3.0 & 0.1 & 0.0 \\ 0.1 & 0.1 & 4.0 & 0.1 \\ 0.1 & 0.0 & 0.1 & 3.0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2.0 \\ 3.0 \\ 4.0 \\ 3.0 \end{bmatrix}$$

This system will be solved by some iterative technique. The A-matrix is not diagonal dominant, thus it is difficult to know whether Jacobi- or Gauss–Seidel iterations will converge. As an alternative, we will try to split the A-matrix so that

$$A = N - P$$

with

$$N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The iteration scheme becomes:

$$N\mathbf{x}^{(k+1)} = P\mathbf{x}^{(k)} + \mathbf{b}.$$

a) Let $\mathbf{x}^{(0)} = [0, 0, 0, 0]^T$, and find $\mathbf{x}^{(2)}$.

b) Prove that the iteration scheme satisfies

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}\|_\infty \leq 0.2 \cdot \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty$$

where $\mathbf{x}$ is the exact solution of (1).

Give an upper limit for the error $\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty$.

Problem 3

a) The value of a function $f(x)$ is known in the points

x_i	0.00	0.60	1.00
$f(x_i)$	0.00	0.75	0.50

Find the polynomial $p(x)$ of lowest possible order interpolating $f(x)$ in these three points.

b) Find the polynomial $q(x)$ of lowest possible order which satisfies the interpolation conditions given above *in addition to* the condition $q'(0.6) = f'(0.6) = 0.5$.

Hint: Let $q(x) = p(x) + r(x)$. What are the zeros of $r(x)$?

c) Prove that under a certain condition (which) exist a $\xi_x \in (0,1)$ so that the interpolation error satisfies

$$f(x) - q(x) = \frac{1}{24} f^{(4)}(\xi_x) w(x),$$

where

$$w(x) = x(x-1)(x-0.6)^2.$$

Problem 4 In this task, you will develop methods for solving second order differential equations on the form

$$y'' = f(t, y(t)), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'. \tag{2}$$

The stepsize h is constant, $t_n = t_0 + nh$ and $y_n \approx y(t_n)$ is the numerical approximation to the solution. You will find some useful information about such equations, and the numerical solution of those in the appendix at the end of this exam set.

a) Write (2) as a system of first order differential equations. Show that two steps of Eulers method can be put together into a two-step method given by

$$y_{n+1} - 2y_n + y_{n-1} = h^2 f(t_{n-1}, y_{n-1}).$$

Discuss the convergence properties of the method.

b) Let $p(t)$ be the polynomial interpolating $f(t, y(t))$ in t_n, t_{n-1} and t_{n-2}. Use this to construct a linear multistep method on the form

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \sum_{l=0}^{2} \beta_l f(t_{n+l-2}, y_{n+l-2}).$$

Find β_0, β_1 and β_2, and find the order of the method.

Hint: Use Newton's Backward difference formula to express the interpolation polynomial $p(t)$.

Appendix

The exact solution of the equation

$$y'' = f(t, y(t)), \qquad y(t_0) = y_0, \qquad y'(t_0) = y'_0 \tag{3}$$

satisfies

$$y(t_{n+1}) - 2y(t_n) + y(t_{n-1}) = \int_{t_n}^{t_{n+1}} (t_n + h - t) f(t, y(t)) dt - \int_{t_{n-1}}^{t_n} (t_n - h - t) f(t, y(t)) dt,$$

with $t_n = t_0 + nh$.

A general linear multistep method applied to (3) is given by

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h^2 \sum_{l=0}^{k} \beta_l f(t_{n+l}, y_{n+l}). \tag{4}$$

The local truncation error is defined by

$$h^2 \tau_{n+k}(h) = \sum_{l=0}^{k} \alpha_l y(t_{n+l}) - h^2 \sum_{l=0}^{k} \beta_l f(t_{n+l}, y(t_{n+l})).$$

For all sufficiently smooth $y(t)$ we can find the Taylor-expansion

$$h^2 \tau_{n+k}(h) = C_0 y(t_n) + C_1 h y'(t_n) + \cdots + C_q h^q y^{(q)}(t_n) + \cdots$$

The method is consistent if

$$\lim_{h \to 0} \tau_{n+k}(h) = 0,$$

and *consistent of order p* if

$$C_0 = C_1 = \cdots = C_{p+1} = 0, \; C_{p+2} \neq 0.$$

It can be proved that the numerical solution is *convergent of order p* if

- The method is consistent of order p.
- It is stable, that is the roots r_j, $j = 1, \ldots, k$ of the characteristic polynomial $\sum_{l=0}^{k} \alpha_l r^l$ all satisfy $|r_j| \leq 1$. If a root satisfies $|r_j| = 1$, the multiplicity of that root can be at most 2.

We will also require sufficient accurate initial values y_l, $l = 0, 1, \ldots, k-1$. For this exam, you can assume this to be satisfied.