1 Bernoulli polynomials and the Euler-Maclaurin formula.

Let the integral

$$I(f) = \int_a^b f(x)dx$$

be approximated by the Trapezoidal rule

$$T(h) = h\left(\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n)\right)$$

where $x_i = a + ih$, $i = 0, \cdots, n$ with $h = (b-a)/n$. The aim of this note is to prove that the error can be written as

$$I(f) - T(h) = \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} h^{2k}(f^{(2k-1)}(a) - f^{(2k-1)}(b)) - \frac{b_{2m}}{(2m)!} h^{2m}(b-a)\, f^{(2m)}(\eta) \tag{1}$$

where $\eta \in (a,b)$, and b_k are the Bernoulli numbers. These will be defined later. Obviously, the formula requires $f \in C^{2m}[a,b]$. The formula proves that the error can be written as an even power series of h, which is the theoretical fundament for the development of the Romberg integration algorithm.

We will first define the well known Bernoulli polynomials which will then be used to prove Euler-Maclaurin's formula. This again will be used to prove (1).

Bernoulli polynomials.

For our purpose, it is convenient to define the Bernoulli polynomials by the following reccurence relation:

$$\begin{aligned} &B_0(t) = 1, \\ &B_k'(t) = kB_{k-1}(t) \quad \text{and} \quad \int_0^1 B_k(t) = 0, \qquad k \geq 1. \end{aligned}$$

The first few polynomials become

$$\begin{aligned} B_1(t) &= t - \frac{1}{2} \\ B_2(t) &= t^2 - t + 1/6 \\ B_3(t) &= t^3 - \frac{3}{2}t^2 + \frac{1}{2}t. \end{aligned}$$

We will need the following properties of these polynomials:

$$B_k(0) = B_k(1), \qquad k \geq 2, \tag{2a}$$

$$B_k(1-t) = (-1)^k B_k(t), \tag{2b}$$

$$B_k(t) - B_k(0) \qquad \text{has no zeros in (0,1) if } k \text{ is even.} \tag{2c}$$

The *Bernoulli numbers* are given by $b_k = B_k(0)$. As a consequence of (2a) and (2b) we get $b_k = 0$ for k odd and $k \geq 3$. The Bernoulli numbers can also be found by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{b_k}{k!} t^k.$$

Euler-Maclaurin's formula

By repeated use of integration by parts and the Bernoulli polynomials, we get

$$\begin{aligned}
\int_0^1 f(t)dt &= \int_0^1 f(t)B_0(t)dt \\
&= f(t)B_1(t)\Big|_0^1 - \int_0^1 B_1(t)f'(t)dt \\
&= \frac{1}{2}[f(1) + f(0)] - \frac{1}{2}B_2(t)f'(t)\Big|_0^1 + \frac{1}{2}\int_0^1 B_2(t)f''(t)dt \\
&= \frac{1}{2}[f(1) + f(0)] - \sum_{j=2}^{\bar{m}} \frac{(-1)^j}{j!} B_j(t) f^{(j-1)}(t)\Big|_0^1 + (-1)^{\bar{m}} \frac{1}{\bar{m}!}\int_0^1 B_n(t)f^{(n)}(t)dt.
\end{aligned}$$

Now, since $B_{2k+1}(0) = B_{2k+1}(1) = 0$ and $B_{2k}(0) = B_{2k}(1) = b_{2k}$ this can be written as

$$\begin{aligned}
\int_0^1 f(t)dt = &\frac{1}{2}[f(1) + f(0)] - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right] \\
&- \frac{b_{2m}}{(2m)!}\left[f^{(2m-1)}(1) - f^{(2m-1)}(0)\right] + \frac{1}{(2m)!}\int_0^1 B_{2m}(t)f^{(2m)}(t)dt.
\end{aligned}$$

Using property (2c) and the mean value theorem for integrals, the last two terms become

$$-\frac{1}{(2m)!}\int_0^1 \left[b_{2m} - B_{2m}(t)\right] f^{(2m)}(t)dt = -\frac{b_{2m}}{(2m)!} f^{(2m)}(\eta).$$

As a result, we get the *Euler-Maclaurin formula*:

$$\int_0^1 f(t)dt = \frac{1}{2}[f(1)+f(0)] - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right] - \frac{b_{2m}}{(2m)!} f^{(2m)}(\eta).$$

Error of the Trapezoidal rule.

Applying the Euler-Maclaurin formula to a function $g(t) = f(x_i + ht)$, using the change of variables $t = (x - x_i)/h$ gives

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} f(x)dx = &\frac{h}{2}[f(x_i) + f(x_{i+1})] - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2k)!} h^{2k}\left[f^{(2k-1)}(x_{i+1}) - f^{(2k-1)}(x_i)\right] \\
&- \frac{b_{2m}}{(2m)!} h^{2m} f^{(2m)}(\eta_i).
\end{aligned}$$

The expression (1) is finally obtained by summing over all the intervals $[x_i, x_{i+1}]$.